Solution to the written test for examination in MVE135
Random processes with applications, 2008-10-23, 14:00 - 18:00, a house on Hörsalsvägen.

There are 30 total points in the examination. One needs 14 points for grade 3 (to pass), 18 points for grade 4, and 24 points for grade 5.

Problem 1. A communication channel accepts an arbitrary voltage input V and outputs a voltage

$$Y = V + N$$

where N is a Gaussian random variable with mean zero and variance one, independent of the input value. Suppose that the channel is used to transmit binary information as follows:

to transmit 0:	input -1
to transmit 1:	input 1.

The receiver decides a 0 was sent if the voltage is negative and a 1 otherwise. Find the probability of the receiver making an error if both inputs are equally probable. 4p

Solution.

$$P\{error|V=-1\} = P\{Y \geq 0|V=-1\} = P\{-1+N \geq 0\} = P\{N \geq 1\} = Q(1) = 0.159$$
$$P\{error|V=1\} = P\{Y < 0|V=1\} = P\{1+N < 0\} = P\{N < -1\} = Q(1) = 0.159.$$

By the total probability formula

$$P\{error\} = P\{error|V=-1\}P\{V=-1\} + P\{error|V=1\}P\{V=1\} = 0.159$$

Problem 2. The number of bytes in a message is described by a random variable N with $P(N=n) = (1-p)p^n, \quad n \geq 0$. The messages are broken into packets of length M bytes. Let Q be the number of full packets in a message and R be the number of bytes left over.

(a) Compute the joint probability mass function of Q and R and the marginal probability mass functions of Q and R. 2p

(b) What is the expected value of Q? Are Q and R independent? 2p

Solution. The joint PMF of Q and R is

(a)

$$P(Q=q,\, R=r) = P\{N = qM + r\} = (1-p)p^{qM+r},$$
$$\text{where} \quad q = 0, 1, \ldots \quad \text{and} \quad r = 0, \ldots M-1.$$

The marginal PMFs are obtained from the joint PMF as

$$P\{Q=q\} = (1-p)p^{qM}\sum_{r=0}^{M-1} p^r = (1-p^M)\big(p^M\big)^q, \quad q = 0, 1, \ldots$$
$$P\{R=r\} = (1-p)p^r\sum_{q=0}^{\infty}(p^M)^q = \frac{(1-p)p^r}{1-p^M}, \quad r = 0, \ldots M-1.$$

(b) Clearly, Q is the geometric random variable and $E[Q] = \dfrac{p^M}{1-p^M}$. Since

$$P\{Q=q,\, R=r\} = P\{Q=q\}P\{R=r\} \quad \text{for} \qquad q=0,\,1,\,\ldots \quad \text{and} \quad r=0,\,\ldots\, M-1$$

Q and R are independent.

Problem 3. The random variables X and Y are jointly Gaussian with expectation 0, variance 1, and correlation coefficient $\frac{1}{4}$. Find the distribution of $Z = X - aY$, where a is some non-zero constant. For which value of a is the variance of Z equal to 1? For this value, compute $E[Z|Z<1]$. 4p

Solution. A linear combination of jointly Gaussian random variables is a Gaussian random variable. Thus Z is Gaussian with

$$E[Z] = E[X] - E[aY] = 0, \quad Var(Z) = 1 + a^2 - \frac{a}{2}.$$

$Var(Z)$ equals 1 when $a(a-\frac{1}{2}) = 0$ or $a = \frac{1}{2}$. In this case Z is a standard normal random variable and we have

$$f_Z(z|Z<1) = \frac{f_Z(z)}{P\{Z<1\}}\,[1-u(z-1)] = \frac{\exp\{-z^2/2\}}{[1-Q(1)]\sqrt{2\pi}}\,[1-u(z-1)]$$

Thus

$$\begin{aligned} E[Z|Z<1] &= \frac{1}{[1-Q(1)]\sqrt{2\pi}} \int_{-\infty}^{1} z\, \exp\{-z^2/2\}\, dz \\ &= \frac{1}{[1-Q(1)]\sqrt{2\pi}} \left[-\exp\{-z^2/2\}\Big|_{-\infty}^{1} \right] = -\frac{\exp\{-1/2\}}{[1-Q(1)]\sqrt{2\pi}} \\ E[Z|Z<1] &= -0.2876 \end{aligned}$$

Problem 4. $N(t)$ is the Poisson process with parameter λ. Show that its autocovariance function is $C_{NN}(t_1,\, t_2) = \lambda \min(t_1,\, t_2)$ and compute the autocovariance function of the process $e^{-t/2}N(e^t)$. 5p

Solution. To compute the autocovariance function of $N(t)$ we use the fact that the increments of $N(t)$ are independent and also that $N(0) = 0$. Suppose $t_1 \le t_2$. We have

$$\begin{aligned} C_{NN}(t_1, t_2) &= Cov\big(N(t_1),\, N(t_2)\big) = Cov\big(N(t_1) - N(0),\, N(t_2) - N(t_1) + N(t_1)\big) \\ &= Cov\big(N(t_1) - N(0),\, N(t_1)\big) = Var(N(t_1)) = \lambda t_1. \end{aligned}$$

Thus for arbitrary t_1 and t_2

$$C_{NN}(t_1,\, t_2) = \lambda \min(t_1,\, t_2).$$

Denote $X(t) = e^{-t/2}N(e^t)$. We have

$$\begin{aligned} C_{XX}(t_1, t_2) &= Cov\big(e^{-t_1/2}N(e^{t_1}),\, e^{-t_2/2}N(e^{t_2})\big) = e^{-(t_1+t_2)/2} C_{NN}(e^{t_1}, e^{t_2}) \\ &= e^{-(t_1+t_2)/2}\lambda\, \min(e^{t_1}, e^{t_2}) = \lambda e^{-(t_1+t_2)/2 + \min(t_1,t_2)} = \lambda e^{-|t_1-t_2|/2} \end{aligned}$$

Problem 5. The input to a linear time invariant system with impulse response

$$h(t) = \begin{cases} 8\delta(t) + 1, & \text{if } 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$$

is the random process

$$X(t) = \sin(2\pi t + \Theta), \quad -\infty < t < \infty,$$

where Θ is a random variable uniformly distributed over $[0, 2\pi)$. Give a formula for the output process $Y(t)$ and compute the mean function of this process. 4p

Solution. $Y(t)$ is computed as the convolution of the input and the impulse response function.

$$Y(t) = \int_0^1 [\sin(2\pi(t-u)+\Theta)[8\delta(u)+1]du = 8\sin(2\pi t+\Theta), \quad \text{since} \quad \int_0^1 \sin(2\pi(t-u)+\Theta)du = 0.$$

The mean function $m_Y(t)$ is

$$m_Y(t) = m_X(t)\int_0^1 h(t)dt = 0, \quad \text{since} \quad m_X(t) = \frac{1}{2\pi}\int_0^{2\pi} \sin(2\pi t + \theta)d\theta = 0.$$

Problem 6. Let $X(t)$ be a wide sense stationary process with autocorrelation function $R_{XX}(\tau)$. A new process is formed by multiplying $X(t)$ by a carrier to produce

$$Y(t) = X(t)\cos(\omega_0 t + \Theta),$$

where ω_0 is a fixed frequency and Θ is a random variable, which is independent of the process $X(t)$ and uniformly distributed over $[0, 2\pi)$. Compute the power spectral density and the average power of $Y(t)$. 4p

Solution.

$$R_{YY}(t,t+\tau) = E[X(t)X(t+\tau)]E[\cos(\omega_0 t+\Theta)\cos(\omega_0(t+\tau)+\Theta)] = R_{XX}(\tau)\frac{1}{2}\cos(\omega_0\tau) = R_{YY}(\tau).$$

To compute the power spectral density of the process $Y(t)$ we apply above the formula

$$\mathcal{F}\Big\{R_{XX}(\tau)\frac{1}{2}\cos(\omega_0\tau)\Big\} = \mathcal{F}\Big\{R_{XX}(\tau)\Big\} * \mathcal{F}\Big\{\frac{1}{2}\cos(\omega_0\tau)\Big\}$$

According to the table on p. 521

$$\mathcal{F}\{\cos(\omega_0\tau)\} = \frac{1}{2}\Big[\delta\Big(f - \frac{\omega_0}{2\pi}\Big) + \delta\Big(f + \frac{\omega_0}{2\pi}\Big)\Big]$$

Then

$$S_{YY}(f) = \frac{1}{4}\Big[S_{XX}\Big(f - \frac{\omega_0}{2\pi}\Big) + S_{XX}\Big(f + \frac{\omega_0}{2\pi}\Big)\Big]$$

The average power of $Y(t)$ is

$$R_{YY}(0) = \frac{1}{2}R_{XX}(0).$$

Problem 7. $Y[n]$ is an AR(1) process defined as

$$Y[n] = \frac{1}{2}Y[n-1] + e[n],$$

where $e[n]$ is the white-noise process of average power σ^2.

(a) Compute $R_{YY}(m)$, the autocorrelation function of $Y[n]$. 2p

(b) Let $\sigma^2 = 3$. Find the unite impulse response of the filter producing the best predictor of $Y[n]$ from $Y[n-2]$ and $Y[n-3]$. 2p

(c) Give a formula for the estimation error. 1p

Solution.

(a) One way to compute the auotcorrelation function of $Y[n]$ is to multiply both parts of the equation defining this process by $Y[n-k]$, $k = 0, 1, \ldots$ and to take expectation from both sides of the new equations. The gives the following recurent equations for $R_{YY}(k)$:

$$R_{YY}(0) = \frac{1}{2}R_{YY}(1) + \sigma^2$$
$$R_{YY}(k) = \frac{1}{2}R_{YY}(k-1), \quad k \geq 1.$$

From the first two equations

$$R_{YY}(0) = \frac{1}{2}R_{YY}(1) + \sigma^2$$
$$R_{YY}(1) = \frac{1}{2}R_{YY}(0)$$

we get $R_{YY}(0) = \frac{4}{3}\sigma^2$ and then

$$R_{YY}(k) = \frac{4}{3}\sigma^2\left(\frac{1}{2}\right)^{|k|}, \quad k = 0, \pm 1, \ldots.$$

(b) Let $\hat{Y}[n] = z_2 Y[n-2] + z_3 Y[n-3]$ be the best predictor. The desired equations follow from the orthogonality condition $Y[n] - \hat{Y}[n] \perp Y[n-2]$ and $Y[n] - \hat{Y}[n] \perp Y[n-3]$:

$z_2 R_{YY}(0) + z_3 R_{YY}(1) = R_{YY}(2)$

$z_2 R_{YY}(1) + z_3 R_{YY}(0) = R_{YY}(3)$.

With $\sigma^2 = 3$ we have $R_{YY}(k) = 4\left(\frac{1}{2}\right)^{|k|}$ and the above then gives

$4z_2 + 2z_3 = 1$

$2z_2 + 4z_3 = \frac{1}{2}$.

Hence $z_2 = \frac{1}{4}$, $z_3 = 0$ and $\hat{Y}[n] = \frac{1}{4}Y[n-2]$.

(c)

$$e^2 = E[(Y[n] - \hat{Y}[n])^2] = E[Y[n](Y[n] - \hat{Y}[n])] = R_{YY}(0) - \frac{1}{4}R_{YY}(2) = 4 - \frac{1}{4} = 3.75.$$