

## Solutions to Chapter 3 Exercises (Part 2)

### Problem 3.5

a) Since

$$\int_{-\infty}^{+\infty} f_X(x) dx = 1$$

$$c \int_{-\infty}^{+\infty} \frac{1}{x^2 + 4} dx = 1$$

$$\frac{c}{4} \arctan\left(\frac{x}{2}\right) \Big|_{-\infty}^{+\infty} = 1 \rightarrow \frac{c}{4} \left( \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) \right) = 1 \rightarrow c = \frac{4}{\pi}, f_X(x) = \frac{4}{\pi(x^2 + 4)}$$

b)

$$\Pr(X > 2) = \int_2^{+\infty} \frac{4}{\pi(x^2 + 4)} dx = \frac{1}{\pi} \arctan\left(\frac{x}{2}\right) \Big|_2^{+\infty} = \frac{1}{\pi} \left( \frac{\pi}{2} - \frac{\pi}{4} \right) = 0.25$$

c)

$$\Pr(X < 3) = \int_{-\infty}^3 \frac{4}{\pi(x^2 + 4)} dx = \frac{1}{\pi} \arctan\left(\frac{x}{2}\right) \Big|_{-\infty}^3 = \frac{1}{\pi} \left( \arctan \frac{3}{2} + \frac{\pi}{2} \right) = 0.8128$$

d)

$$\begin{aligned} \Pr(X < 3 | X > 2) &= \frac{\Pr(2 < X < 3)}{\Pr(X > 2)} \\ &= \frac{\int_2^3 \frac{4}{\pi(x^2 + 4)} dx}{0.25} \\ &= \frac{\frac{1}{\pi} \arctan\left(\frac{x}{2}\right) \Big|_2^3}{0.25} \\ &= \frac{\frac{1}{\pi} \left( \arctan \frac{3}{2} - \frac{\pi}{4} \right)}{0.25} = 0.2513 \end{aligned}$$

### Problem 3.9

Since

$$\exp(-(ax^2+bx+c)) = \exp\left(-\left(\frac{(x + \frac{b}{2a})^2 + \frac{c}{a} - \frac{b^2}{4a^2}}{\frac{1}{a}}\right)\right) = \exp\left(\frac{b^2}{4a} - c\right) \exp\left(-\frac{(x + \frac{b}{2a})^2}{\frac{1}{a}}\right),$$

then we have

$$\begin{aligned} \int_{-\infty}^{\infty} \exp(-(ax^2 + bx + c)) dx &= \int_{-\infty}^{\infty} \exp\left(\frac{b^2}{4a} - c\right) * \exp\left(-\frac{(x + \frac{b}{2a})^2}{\frac{1}{a}}\right) dx \\ &= \exp\left(\frac{b^2}{4a} - c\right) \sqrt{\frac{\pi}{a}} * \frac{1}{\sqrt{\frac{\pi}{a}}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x + \frac{b}{2a})^2}{\frac{1}{a}}\right) dx. \end{aligned}$$

Since

$$\frac{1}{\sqrt{\frac{\pi}{a}}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x + \frac{b}{2a})^2}{\frac{1}{a}}\right) dx = 1,$$

then

$$\int_{-\infty}^{\infty} \exp(-(ax^2 + bx + c)) dx = \sqrt{\frac{\pi}{a}} \exp\left(\frac{b^2}{4a} - c\right).$$

### Problem 3.14

(a)

$$\begin{aligned} f_{X|X>0}(x) &= \begin{cases} \frac{f_X(x)}{Pr(X>0)} & x > 0 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{f_X(x)}{Q(0)} & x > 0 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 2f_X(x) & x > 0 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{2}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right) & x > 0 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

(b)

$$\begin{aligned} f_{X||X|<3}(x) &= \begin{cases} \frac{f_X(x)}{Pr(-3<X<3)} & -3 < x < 3 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{f_X(x)}{1-2Q(\frac{3}{\sigma})} & -3 < x < 3 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{1}{1-2Q(\frac{3}{\sigma})} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right) & -3 < x < 3 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

(c)

$$\begin{aligned} f_{X||X|>3}(x) &= \begin{cases} \frac{f_X(x)}{Pr(|X|>3)} & |X| > 3 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{f_X(x)}{2Q(\frac{3}{\sigma})} & |X| > 3 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{1}{2Q(\frac{3}{\sigma})} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-\frac{x^2}{2\sigma^2}\right) & |X| > 3 \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

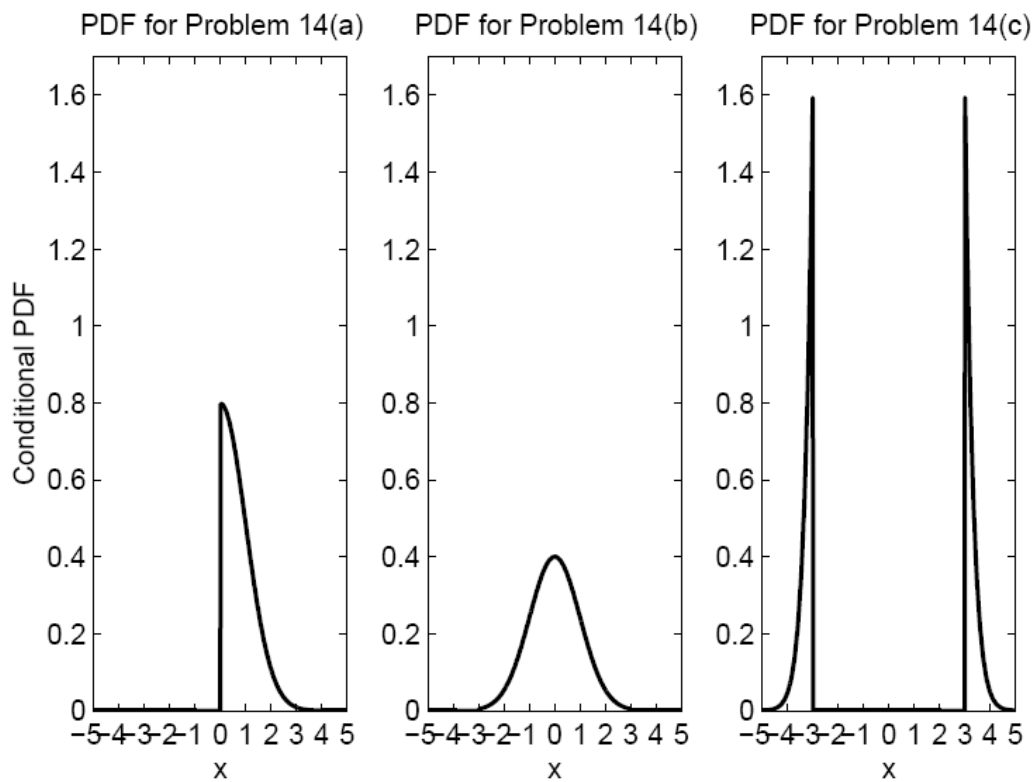


Figure 1 PDF plots for problem 3.14; ( $\sigma^2=1$ ).

### Problem 3.19

```
function Es_Pr=PrEs(x1,x2,sigma,m)

%Es_Pr=PrEs(x1,x2,sigma,m)
%This program will calculate the probability  $\Pr(x1 \leq X \leq x2)$ 
%for X which is Gaussian random variable

%%%%%%%% Inputs:
%%%%%%%% x1:  lower bound
%%%%%%%% x2:  upper bound
%%%%%%%% sigma: standard deviation
%%%%%%%% m:   mean value

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Es_Pr=0.5*erfc(((x1-m)/sigma)/sqrt(2))-0.5*erfc(((x2-m)/sigma)/sqrt(2));
```