RANDOM PROCESSES WITH APPLICATIONS 2009
SOLUTION TO HOMEWORK 1

This assignment is optional. It gives two bonus points to the written examination, when the submitted solution collects 9 points or more out of 16.

Day assigned: September 7
Due date: September 18, 15:15

Problem 1. Three people each write down in a random order and independently the numbers 1, 2, ... 10.

(a) Compute the probability that they all have the same number in the first position. 1p

(b) Give a formula of the probability that there isn't a position with the same number in it. 2p

Hint. Use the inclusion-exclusion formula.

Solution

(a) The problem can be reformulated as follows. We choose equally likely a permutation among all permutations of 1, 2, ... 10. In three independent repetitions of the trial, what is the probability of the same number in the first position? The answer is

$$10 \times \left(\frac{9!}{10!}\right)^3 = \left(\frac{9!}{10!}\right)^2 = 0.01$$

(b) Consider the events

$$A_j = \{\text{same number in position } j\},\ j = 1, 2, \ldots, 10.$$

We have

$$P(A_j) = 0.01,\ j = 1, ... 10;$$
$$P\{k \textit{ fixed events of } A_1, A_2, \ldots, A_{10} \text{ occur}\}$$
$$= 10(10-1)...(10-k+1) \times \left(\frac{(10-k)!}{10!}\right)^3 = \left(\frac{(10-k)!}{10!}\right)^2.$$

By the inclusion-exclusion formula

$$P(A_1 \cup A_2 \ldots \cup A_{10}) = \sum_1^{10} (-1)^{k-1} \binom{10}{k} \left(\frac{(10-k)!}{10!}\right)^2$$
$$= \sum_1^{10} (-1)^{k-1} \frac{1}{k!} \left(\frac{(10-k)!}{10!}\right).$$

The desired probability is then

$$P(\overline{A_1 \cup A_2 \ldots \cup A_{10}}) = 1 - P(A_1 \cup A_2 \ldots \cup A_{10})$$
$$= 1 - \sum_{1}^{10} (-1)^{k-1} \frac{1}{k!} \frac{(10-k)!}{10!}.$$

Problem 2. A binary information source (e.g., a fax machine) generates long strings of 0_s followed by 1, where the symbols are independent and $P\{\text{symbol} = 0\} = (1/2)^{1/M}$. Here $M = 2^m$ and m is a fixed positive integer. Let the random variable X denote the length of the runs of 0_s between consecutive 1_s. This length is encoded as follows. Let $X = n$.

- Express n as a multiple of M and a reminder r, that is, find k and r such that

$$n = kM + r, \quad 0 \leq r \leq M - 1.$$

- Encode the value n into a codeword that contains a prefix consisting of k 0_s followed by 1, and a suffix consisting of the m-bit representation of the reminder r.
  Reminder. Existence of the m-bit representation of the reminder can be seen from the fact that there is one-to-one mapping between the 2^m different values of the reminder and the 2^m different binary vectors of length m.

From the binary string obtained the decoder can deduce the value n.

(a) Let the random variable K denote the length of the prefix. Find the distribution of K. 1p

(b) Let the random variable N denote the length of a codeword. Find $E[N]$. 1p

(c) Find the compression ratio, which is defined as the ratio of the average length of runs including the final 1 and the average length of a codeword. 1p

Solution
Note first that X is a geometric rv with range $S_X = \{0, 1, \ldots\}$ and parameter

$$p = 1 - P\{\text{symbol} = 0\} = 1 - \left(\frac{1}{2}\right)^{1/M}.$$

(a) The range of K is $S_K = \{0, 1, \ldots\}$. We have

$$P\{K = k\} = \sum_{r=0}^{M-1} P\{X = kM + r\} = \sum_{r=0}^{M-1} (1-p)^{kM+r} p$$
$$= \left(\frac{1}{2}\right)^k \left[1 - \left(\frac{1}{2}\right)^{1/M}\right] (\sum_{r=0}^{M-1} \left(\frac{1}{2}\right)^{r/M} = \left(\frac{1}{2}\right)^{k+1}.$$

Thus K is a geometric random variable with range $S_K = \{0, 1, \ldots\}$ and parameter $\frac{1}{2}$.

(b)

$$E[N] = E[K] + 1 + m = m + 2$$

(c)

$$E[X+1] = \frac{1-p}{p} + 1 = \frac{1}{p} = \frac{1}{1-\left(\frac{1}{2}\right)^{1/M}}$$

$$\frac{E[X+1]}{E[N]} = \frac{1}{(m+2)\left[1-\left(\frac{1}{2}\right)^{1/M}\right]}.$$

Problem 3. A binary transmission channel introdices independent bit errors with probability 0.15. Estimate the probability that there are 20 or fewer errors in 100 bit transmissions. 2p
Hint. Use the Central Limit Theorem.

Solution

The number X of errors in one bit transmission is a Bernoulli random variable with $E[X] = 0.15$ and $Var(X) = 0.15 \cdot 0.85 = 0.1275$. The total number of errors in 100 bit transmissions is a sum of 100 independent copies of X

$$S_{100} = X_1 + X_2 + \ldots + X_{100}.$$

By the Central limit Theorem

$$P\{S_{100} \leq 20\} = \Phi\left(\frac{20 - 100 \times 0.15}{10 \times \sqrt{0.1275}}\right) = \Phi(1.4) = 1 - Q(1.4) = 0.92$$

Problem 4. Two random variables X and Y are jointly Gaussian with mean vector and covariance matrix given by

$$\mathbf{m} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \textit{and} \quad \mathbf{C} = \begin{bmatrix} 4 & -4 \\ -4 & 9 \end{bmatrix}.$$

respectively.

(a) Find the correlation coefficient between X and Y. 1p

(b) Use matrix operations to find the covariance function of $Z = 2X + Y$ and $W = X - 2Y$. 2p

(c) Find the PDF of Z. 1p

Solution

(a)

$$\rho_{X,Y} = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}} == \frac{-4}{2.3} = -\frac{2}{3}.$$

(b) From

$$\begin{bmatrix} Z \\ W \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix}$$

we obtain

$$\mathbf{C}_{Z,W} = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 4 & -4 \\ -4 & 9 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 9 & 2 \\ 2 & 56 \end{bmatrix}$$

(c) Z is Gaussian with mean 4 and variance 9.

Problem 5. The random variables X and Y are sample of a random signal at two time instants. Suppose they are independent and zero-mean Gaussian with the same variance. When signal "0" is present the variance is σ_0^2; when signal "1" is present the variance is $\sigma_1^2 > \sigma_0^2$. Suppose signals "0" and "1" occur with probabilities p and $1-p$, respectively. Let $R^2 = X^2 + Y^2$ be the total energy of the observations.

(a) Find the PDF of R^2 2p

(b) Suppose we use the following "signal detection" rule: If $R^2 > T$, then we decide signal "1" is present; otherwise we decide signal "0" is present. Find an expression for the probability of detection error in terms of T. 2p

Solution

(a) For $i = 0, 1$ let

$$R_i^2 = \frac{R^2}{\sigma_i^2} = \frac{X^2}{\sigma_i^2} + \frac{Y^2}{\sigma_i^2}.$$

R_1^2 and R_2^2 are Chi-square distributed with two degrees of freedom with PDF$_s$

$$f_i(u) = \frac{e^{-u/2}}{2}$$

$$\begin{aligned}
F(r) = P\{R^2 \leq r\} &= pP\{R^2 \leq r|''0'' \text{ present}\} + (1-p)P\{R^2 \leq r|''1'' \text{ present}\} \\
&= pP\left\{R_0^2 \leq \frac{r}{\sigma_0^2}|''0'' \text{ present}\right\} + (1-p)P\left\{R_1^2 \leq \frac{r}{\sigma_1^2}|''1'' \text{ present}\right\} \\
&= pF_0\left(\frac{r}{\sigma_0^2}\right) + (1-p)F_1\left(\frac{r}{\sigma_1^2}\right).
\end{aligned}$$

Thus

$$\begin{aligned}
f(r) = \frac{dF(r)}{dr} &= \frac{p}{\sigma_0^2} f_0\left(\frac{r}{\sigma_0^2}\right) + \frac{1-p}{\sigma_1^2} f_1\left(\frac{r}{\sigma_1^2}\right) \\
&= \frac{p}{2\sigma_0^2} e^{-r/2\sigma_0^2} + \frac{1-p}{2\sigma_1^2} e^{-r/2\sigma_1^2}.
\end{aligned}$$

(b)

$$\begin{aligned}
P_{error} &= pP\{R^2 > T|''0'' \text{ present}\} + (1-p)P\{R^2 < T|''1'' \text{ present}\} \\
& pP\{R^2/\sigma_0^2 > T/\sigma_0^2|''0'' \text{ present}\} + (1-p)P\{R^2/\sigma_1^2 < T/\sigma_1^2|''1'' \text{ present}\} \\
& pP\{R_0^2 > T/\sigma_0^2\} + (1-p)P\{R_1^2 < T/\sigma_1^2\} \\
&= p\int_{T/\sigma_0^2}^{\infty} f_0(u)du + (1-p)\int_0^{T/\sigma_1^2} f_1(u)du \\
&= pe^{-\frac{T}{2\sigma_0^2}} + (1-p)\left[1 - e^{-\frac{T}{2\sigma_1^2}}\right].
\end{aligned}$$