RANDOM PROCESSES WITH APPLICATIONS 2009
SOLUTION TO HOMEWORK 2

This assignment is optional. It gives two bonus points to the written examination, when the submitted solution collects 12 points or more out of a total of 16 points.

Day assigned: September 28
Due date: October 5, 15:15

Problem 1. Let X be the number of active speakers in a group of M independent speakers, each one of which is active with probability p. Suppose that a voice transmission system can transmit up to $N < M$ voice signals at a time, and that when X exceeds N, $X - N$ randomly selected signals are discarded. Give a formula for computing the expected value of the discarded voices. Estimate the probability that voices are not discarded if $M = 45$, $p = 1/3$, $N = 16$. (2)

Solution. We have $X \sim$ is Binomial (M, p)

Let Y denote the number of discarded voices. Then

$$Y = \begin{cases} 0 & \text{if} \quad X \leq N, \\ X - N & \text{if} \quad X > N \end{cases}$$

and

$$E[Y] = \sum_{j=1}^{M-N} jP\{X = N + j\} = \sum_{j=1}^{M-N} j \binom{M}{N+j} p^{N+j}(1-p)^{M-N-j}.$$

$$P\{Y = 0\} = P\{X \leq N\} \approx \Phi\Big(\frac{N - M \cdot p}{\sqrt{Mp(1-p)}}\Big) = \Phi\Big(\frac{16 - 45 \cdot 1/3}{\sqrt{45 \cdot 1/3 \cdot 2/3}}\Big) = 1 - Q(0.31) = 0.62$$

Problem 2. The joint PDF of the random variables Z_1 and Z_2 is given by

$$f_{Z_1, Z_2}(z_1, z_2) = \frac{1}{\sqrt{2}\pi} e^{-(z_1^2 - \sqrt{2} z_1 z_2 + z_2^2)}.$$

Compute $Cov\,(Z_1 - Z_2/\sqrt{2},\ Z_2)$. (2)

Solution. The power of the exponent in the joint PDF can be written as

$$-\frac{1}{2[1 - (1/\sqrt{2})^2]} \left[z_1^2 + z_2^2 - 2\frac{1}{\sqrt{2}}\, z_1\, z_2 \right].$$

Thus Z_1 and Z_2 are jointly Gaussian with

$$\rho\,(Z_1, Z_2) = \frac{1}{\sqrt{2}}, \quad E\,[Z_i] = 0, \quad Var\,(Z_i) = 1, \quad i = 1, 2,$$

and

$$\text{Cov}\,(Z_1 - Z_2/\sqrt{2},\ Z_2) = \text{Cov}\,(Z_1, Z_2) - \frac{1}{\sqrt{2}} \text{Cov}\,(Z_2, Z_2) = 0.$$

Problem 3.

Problem 3. Let $N(t)$ be a Poisson process with parameter α. Suppose each time an event occurs, independently a coin is flipped and the outcome (heads or tails) is recorded. Let $N_1(t)$ and $N_2(t)$ denote the number of heads and tails recorded up to time t, respectively. Assume that p is the probability of heads.

(a) Compute the conditional probability

$$P\{N_1(t) = m,\ N_2(t) = n - m \,|\, N(t) = n\},$$

where n and m are non-negative integers satisfying $0 \le m \le n$. 2p.

(b) Compute the joint PMF of $N_1(t)$ and $N_2(t)$. 2p.

(c) Compute the PMF of $N_1(t)$ and the PMF of $N_2(t)$. Are $N_1(t)$ and $N_2(t)$ independent? 2p

Solution.

(a) Since the process $N(t)$ and the coin tosses are independent we have

$$P\{N_1(t) = m,\ N_2(t) = n - m \,|\, N(t) = n\} = \binom{n}{m} p^m (1-p)^{n-m}.$$

(b)

$$\begin{aligned} P\{N_1(t) = m,\ N_2(t) = k\} &= P\{N_1(t) = m,\ N_2(t) = k \,|\, N(t) = m + k\} P\{N(t) = m + k\} \\ &= \binom{m+k}{m} p^m (1-p)^{n-m} \cdot \frac{(\lambda t)^{m+k}}{(m+k)!} e^{-\lambda t} \\ &= \frac{(\lambda t p)^m}{m!} e^{-\lambda t p} \cdot \frac{(\lambda t(1-p))^k}{k!} e^{-\lambda t(1-p)}. \end{aligned}$$

(c) The joint PMF $P_{N_1(t),\, N_2(t)}(m,\, k)$ above is a product of two PMFs

$$P_1(m) \sim Poisson(\lambda t p) \text{ and } P_2(k) \sim Poisson(\lambda t(1-p)).$$

Hence $N_1(t)$ and $N_2(t)$ are independent and

$$N_1(t) \sim Poisson(\lambda t p) \text{ and } N_2(t) \sim Poisson(\lambda t(1-p)).$$

Problem 4. Consider the random process

$$Y(t) = (-1)^{X(t)}$$

where $X(t)$ is a Poisson process with rate λ.

(a) Find the mean-value time function and the autocorrelation function of $Y(t)$. Is the process WSS? (3)

(b) Consider the process $Z(t) = AY(t)$, where A is a random variable, independent of $Y(t)$ and with equally likely values ± 1. Is $Z(t)$ WSS? Find the power spectral density of $Z(t)$. (3)

Solution. Denote $p_e(t) = P\{X(t) \text{ is even }\}$. We know

$$p_e(t) = \frac{1 + e^{-2\lambda t}}{2}.$$

(a) Clearly $Y(0) = 1$ and for $t > 0$ we have

$$P\{Y(t) = 1\} = p_e(t), \quad P\{Y(t) = -1\} = 1 - p_e(t)$$

$$E[Y(t)] = 1 \times p_e(t) + (-1) \times (1 - p_e(t) = e^{-2\lambda t}$$

Thus

$$m_Y(0) = 1 \quad \text{and} \quad m_Y(t) = e^{-2\lambda t}, \quad t > 0$$

For $t \geq 0$ and $\tau \geq 0$ we have

$$R_Y(t,\ t+\tau) = E[Y(t)Y(t+\tau)] = 1 \cdot p_e(\tau) + (-1) \cdot (1 - p_e(\tau)) = e^{-2\lambda\tau}$$

Hence

$$R_Y(\tau) = e^{-2\lambda|\tau|}$$

$Y(t)$ is not WSS, since $m_Y(t)$ is not a constant. The process is known as the *semirandom telegraph signal* because its initial value $Y(0)$ is not random.

(b)

$$\mu_Z(t) = E[Z(t)] = E[A]E[Y(t)] = 0$$

$$R_Z(t, t+\tau) = E[A^2]E[Y(t)Y(t+\tau)] = R_Y(\tau) = e^{-2\lambda|\tau|}$$

$Z(t)$is WSS. In fact, $Z(t)$ is the random telegraph signal process. Taking the Fourier transform of $R_z(\tau)$ we obtain

$$S_Z(f) = \frac{4\lambda}{4\lambda + 4\pi^2 f^2}$$