Solutions to Chapter 10 Exercises (Part 2)

Problem 10.5

$$S_{X,X}(f) = FT[R_{X,X}(\tau)] = FT[1] = \delta(f).$$

That is, all power in the process is at d.c.

Problem 10.7

$$\begin{aligned}
R_{X,X}(t,t+\tau) &= E[b^2\cos(\omega t+\Theta)\cos(\omega(t+\tau)+\Theta)] \\
&= \frac{b^2}{2}\cos(\omega\tau) + \frac{b^2}{2}E[\cos(\omega(2t+\tau)+2\Theta)] \\
R_{X,X}(\tau) &= \langle R_{X,X}(t,t+\tau)\rangle = \frac{b^2}{2}\cos(\omega\tau) \\
S_{X,X}(f') &= \frac{b^2}{4}\delta(f'-f) + \frac{b^2}{4}\delta(f'+f)
\end{aligned}$$

This PSD is independent of the distribution of Θ. This is expected because the process has all its power at frequency, f, regardless of the phase Θ.

Problem 10.10

(a)

$$R_{Z,Z}[k] = R_{X,X}[k] + R_{Y,Y}[k] = \left(\frac{1}{2}\right)^{|k|} + \left(\frac{1}{3}\right)^{|k|}$$

(see Exercise 8.18 for details)

(b) For a funcion of the form $R[k] = p^{|k|}$, the Fourier Transform is (t_o is the time between samples of the discrete time process)

$$\begin{aligned}
S(f) &= \sum_k R[k]e^{-j2\pi kft_o} \\
&= 1 + \sum_{k=1}^{\infty} p^k\{e^{-j2\pi kft_o} + e^{j2\pi kft_o}\} \\
&= 1 + \frac{pe^{-j2\pi kft_o}}{1-pe^{-j2\pi kft_o}} + \frac{pe^{j2\pi kft_o}}{1-pe^{j2\pi kft_o}} \\
&= \frac{1-p^2}{1+p^2-2p\cos(2\pi ft_o)}
\end{aligned}$$

Therefore,

$$\begin{aligned}
S_{X,X}(f) &= \frac{3/4}{5/4-\cos(2\pi ft_o)} \\
S_{Y,Y}(f) &= \frac{8/9}{10/9-(2/3)\cos(2\pi ft_o)} \\
S_{Z,Z}(f) &= S_{X,X}(f) + S_{Y,Y}(f).
\end{aligned}$$

Problem 10.17

$$X[n] = \frac{1}{2}X[n-1] + E[n]. \qquad (1)$$

Taking expectations of both sides of (1) results in

$$\mu[n] = \frac{1}{2}\mu[n-1], \qquad n = 1, 2, 3, \ldots.$$

Hence $\mu[n] = (1/2)^n \mu[0]$. Noting that $X(0) = 0$, then $\mu[0] = 0 \Rightarrow \mu[n] = 0$. Multiply both sides of (1) by $X[k]$ and then take expected values to produce

$$E[X[k]X[n]] = \frac{1}{2}E[X[k]X[n-1]] + E[X[k]E[n]].$$

Assuming $k < n$, $X[k]$ and $E[n]$ are independent. Thus, $E[X[k]E[n]] = 0$ and therefore

$$\begin{aligned} R_{X,X}[k,n] &= \frac{1}{2}R_{X,X}[k,n-1]. \\ \Rightarrow R_{X,X}[k,n] &= \left(\frac{1}{2}\right)^{n-k} R_{X,X}[k,k], \qquad n = k, k+1, k+2, \ldots. \end{aligned}$$

Following a similar procedure, it can be shown that if $k > n$

$$R_{X,X}[k,n] = \left(\frac{1}{2}\right)^{k-n} R_{X,X}[k,k].$$

Hence in general

$$R_{X,X}[k,n] = \left(\frac{1}{2}\right)^{|n-k|} R_{X,X}[m,m], \text{ where } m = \min(n,k).$$

Note that $R_{X,X}[m,m]$ can be found as follows:

$$\begin{aligned} R_{X,X}[m,m] &= E[X^2[m]] = E[(\frac{1}{2}X[m-1] + E[m])^2] \\ &= \frac{1}{4}R_{X,X}[m-1,m-1] + E[X[m-1]E[m]] + E[E^2[m]]. \end{aligned}$$

Since $X[m-1]$ and $E[m]$ are uncorrelated, we have the following recursion

$$\begin{aligned} R_{X,X}[m,m] &= \frac{1}{4}R_{X,X}[m-1,m-1] + \sigma_E^2 \\ \Rightarrow R_{X,X}[m,m] &= \left(\frac{1}{4}\right)^m R_{X,X}[0,0] + \sigma_E^2 \sum_{i=0}^{m-1} \left(\frac{1}{4}\right)^i. \end{aligned}$$

Note that since $X(0) = 0$, $R_{X,X}(0,0) = 0$. Therefore

$$\begin{aligned} R_{X,X}[m,m] &= \sigma_E^2 \frac{1-(1/4)^m}{1-1/4} = \frac{4\sigma_E^2}{3}(1-(1/4)^m) \\ \Rightarrow R_{X,X}[k,n] &= \frac{4\sigma_E^2}{3}(1-(1/4)^m)\left(\frac{1}{2}\right)^{|n-k|}. \end{aligned}$$

Since $m = \min(n,k)$ is not a function of $n-k$, the process is not WSS.

Problem 10.20

(a)

$$\begin{aligned} E[\epsilon^2] &= E[(Y[n+1] - \sum_{k=1}^{p} a_k Y[n-k+1])^2] \\ &= E[Y^2[n+1]] - 2\sum_{k=1}^{p} a_k E[Y[n+1]Y[n+1-k]] \\ &+ \sum_{k=1}^{p}\sum_{m=1}^{p} a_k a_m E[Y[n+1-k]Y[n+1-m]] \\ &= R_{Y,Y}[0] - 2\sum_{k=1}^{p} a_k R_{Y,Y}[k] + \sum_{k=1}^{p}\sum_{m=1}^{p} a_k a_m R_{Y,Y}[m-k] \end{aligned}$$

To simplify the notation, introduce the following vectors and matrices:

$$\begin{aligned} \mathbf{r} &= [\,R_{Y,Y}[1] \quad R_{Y,Y}[2] \quad \ldots \quad R_{Y,Y}[p]\,]^T, \\ \mathbf{a} &= [\,a_1 \quad a_2 \quad \ldots \quad a_p\,]^T, \\ \mathbf{R} &= p \text{ x } p \text{ matrix whose } (k,m)\text{th element is } R_{Y,Y}[m-k]. \end{aligned}$$

Then the mean squared error is

$$E[\epsilon^2] = R_{Y,Y}[0] - 2\mathbf{r}^T\mathbf{a} + \mathbf{a}^T\mathbf{R}\mathbf{a}.$$

(b)

$$\begin{aligned} \nabla_{\mathbf{a}} &= -2\mathbf{r} + 2\mathbf{R}\mathbf{a} = 0 \\ \Rightarrow \mathbf{a} &= \mathbf{R}^{-1}\mathbf{r} \end{aligned}$$