Solutions to Chapter 2 Exercises (Part 2)

Problem 2.8

Let A_i = ith dart thrown hits target.
Method 1:

$$\begin{aligned}
Pr(\text{at least one hit out of 3 throws}) &= Pr(A_1 \cup A_2 \cup A_3) \\
&= Pr(A_1) + Pr(A_2) + Pr(A_3) - Pr(A_1 \cap A_2) \\
&- Pr(A_1 \cap A_3) - Pr(A_2 \cap A_3) \\
&+ Pr(A_1 \cap A_2 \cap A_3).
\end{aligned}$$

$Pr(A_i) = 1/4$, $Pr(A_i \cap A_j) = 1/16$, and $Pr(A_i \cap A_j \cap A_k) = 1/64$.

$$Pr(\text{at least one hit}) = 3 \cdot \frac{1}{4} - 3 \cdot \frac{1}{16} + \frac{1}{64} = \frac{37}{64}.$$

Method 2:

$$\begin{aligned}
Pr(\text{at least one hit}) &= 1 - Pr(\text{no hits}) = 1 - Pr(\overline{A_1} \cap \overline{A_2} \cap \overline{A_3}) \\
&= 1 - Pr(\overline{A_1}) \cdot Pr(\overline{A_2}) \cdot Pr(\overline{A_3}) = 1 - \frac{3}{4}^3 = \frac{37}{64}.
\end{aligned}$$

Note 1: To complete this problem, we must assume that the outcome of each throw is independent of all others.
Note 2: The sample space is not equally likely.

Problem 2.12

(a) There are m^n distinct words.
(b) For this case there are 4^3 distinct words.

$$Pr(\text{2 pulses of level 2}) = \binom{4}{2} \cdot \left(\frac{1}{3}\right)^2 \cdot \left(\frac{2}{3}\right)^{4-2} = \frac{8}{27}.$$

Problem 2.19

(a)

$$\begin{aligned} Pr\,(1 \text{ Heart}) &= 13 \cdot Pr\left(H, \overline{H}, \overline{H}, \ldots, \overline{H}\right) \\ &= 13 \cdot \frac{13}{52} \cdot \frac{39}{51} \cdot \frac{38}{50} \cdots \frac{28}{40} = \frac{\binom{13}{1} \cdot \binom{39}{12}}{\binom{52}{13}} = 0.08006 \end{aligned}$$

(b)

We can choose anywhere between 7 to 13 cards of a given suit to satisfy the given condition. All these events are mutually exclusive. Let A_i denote the event of having i Hearts. Following a procedure similar to part (a), it is found that

$$Pr\,(A_i) = \frac{\binom{13}{i} \cdot \binom{39}{13-i}}{\binom{52}{13}}.$$

Therefore

$$\begin{aligned} Pr\,(\text{at least 7 Hearts}) = Pr\left(\bigcup_{i=7}^{13} A_i\right) &= \sum_{i=7}^{13} Pr(A_i) \\ = Pr\left(\bigcup_{i=7}^{13} A_i\right) &= \sum_{i=7}^{13} \frac{\binom{13}{i} \cdot \binom{39}{13-i}}{\binom{52}{13}}. \end{aligned}$$

The probability of at least 7 cards from any suit is simply 4 times the probability of at least 7 Hearts. Hence

$$Pr\,(\text{at least 7 cards from any suit}) = \frac{4}{\binom{52}{13}} \sum_{i=7}^{13} \binom{13}{i} \cdot \binom{39}{13-i} = 0.0403.$$

(c)

$$\begin{aligned} Pr\,(\text{no Hearts}) &= Pr\left(\overline{H}, \overline{H}, \overline{H}, \ldots, \overline{H}\right) \\ &= \frac{39}{52} \cdot \frac{38}{51} \cdot \frac{37}{50} \cdots \frac{27}{40} = \frac{\binom{39}{13}}{\binom{52}{13}} = 0.01279 \end{aligned}$$

Problem 2.22

$$Pr(\text{all } n \text{ versions erased}) = \left(\frac{1}{2}\right)^n$$

Problem 2.25

(a) Since

$$\sum_{k=0}^{\infty} P_X(k) = 1 \ ,$$

i.e.,

$$c \sum_{k=0}^{\infty} 0.37^k = 1 \ .$$

Hence $\frac{c}{1-0.37} = 1$, which gives $c = 0.63$.
(b) Similarly,

$$c \sum_{k=1}^{\infty} 0.82^k = 1 \ .$$

Hence $\frac{c}{1-0.82} = 1 + c$, which gives $c = 0.2195$.
(c) Similarly,

$$c \sum_{k=0}^{24} 0.41^k = 1$$

which gives $c = 0.5900$.
(d) Similarly,

$$c \sum_{k=1}^{15} 0.91^k = 1$$

which gives $c = 0.1307$.
(e) Similarly,

$$c \sum_{k=0}^{6} 0.41^{2k} = 1$$

which gives $c = 0.8319$.

Problem 2.32

(a) If more than 1 error occurs in a 7-bit data block, the decoder will be in error. Thus, the decoder error probability is

$$P_e = \sum_{i=2}^{7} \binom{7}{i} (0.03)^i (1-0.03)^{7-i} = 0.0171 \; .$$

(b) Similarly,

$$P_e = \sum_{i=3}^{15} \binom{15}{i} (0.03)^i (1-0.03)^{15-i} = 0.0094 \; .$$

(c) Similarly,

$$P_e = \sum_{i=4}^{31} \binom{31}{i} (0.03)^i (1-0.03)^{31-i} = 0.0133 \; .$$