Solutions to Chapter 2 Exercises

Problem 2.3

$$
\begin{aligned}
&Pr(A \cup B \cup C) \\
&= Pr((A \cup B) \cup C) \\
&= Pr(A \cup B) + Pr(C) - Pr((A \cup B) \cap C) \\
&= Pr(A) + Pr(B) - Pr(A \cap B) + Pr(C) - Pr((A \cap C) \cup (B \cap C)) \\
&= Pr(A) + Pr(B) + Pr(C) - Pr(A \cap B) - Pr((A \cap C) \cup (B \cap C)) \\
&= Pr(A) + Pr(B) + Pr(C) - Pr(A \cap B) \\
&- (Pr((A \cap C) + Pr(B \cap C)) - Pr(A \cap C \cap B \cap C)) \\
&= Pr(A) + Pr(B) + Pr(C) - Pr(A \cap B) - Pr(A \cap C) - Pr(B \cap C) + Pr(A \cap B \cap C)
\end{aligned}
$$

Problem 2.6

Assume S is the sample space for a given experiment.
Axiom 2.1: For an event A in S,

$$P(A) = \lim_{n\to\infty} \frac{n_A}{n} .$$

Since $n_A \geq 0$, and $n > 0$, $P(A) \geq 0$.
Axiom 2.2: S is the sample space for the experiment. Since S must happen with each run of the experiment, $n_S = n$. Hence

$$P(S) = \lim_{n\to\infty} \frac{n_S}{n} = 1 .$$

Axiom 2.3a: Suppose $A \cap B = 0$. For an experiment that is run n times, assume the event $A \cup B$ occurs n' times, while A occurs n_A times and B occurs n_B times. Then we have $n' = n_A + n_B$. Hence

$$P(A \cup B) = \lim_{n\to\infty} \frac{n'}{n} = \lim_{n\to\infty} \frac{n_A + n_B}{n} = \lim_{n\to\infty} \frac{n_A}{n} + \lim_{n\to\infty} \frac{n_B}{n} = P(A) + P(B) .$$

Axiom 3.b: For an experiment that is run n times, assume the event A_i occurs n_{A_i} times, $i = 1, 2, \cdots$. Define event $C = A_1 \cup A_2 \cdots \cup A_i \cdots$. Since any two events are mutually exclusive, event C occurs $\sum_{i=1}^{\infty} n_{A_i}$ times. Hence,

$$P(\bigcup_{i=1}^{\infty} A_i) = \lim_{n\to\infty} \frac{\sum_{i=1}^{\infty} n_{A_i}}{n} = \sum_{i=1}^{\infty} \lim_{n\to\infty} \frac{n_{A_i}}{n} = \sum_{i=1}^{\infty} P(A_i) .$$

Problem 2.10

(a)

$$Pr(1\text{st} = \text{red}, 2\text{nd} = \text{blue}) = Pr(1\text{st} = \text{red}) Pr(2\text{nd} = \text{blue} \mid 1\text{st} = \text{red})$$
$$Pr(1\text{st} = \text{red}) = \frac{3}{12}$$
$$Pr(2\text{nd} = \text{blue} \mid 1\text{st} = \text{red}) = \frac{5}{11}$$
$$Pr(1\text{st} = \text{red}, 2\text{nd} = \text{blue}) = \frac{3}{12} \cdot \frac{5}{11} = \frac{5}{44}.$$

(b) $Pr(2\text{nd} = \text{white}) = \frac{4}{12} = \frac{1}{3}$.
(c)

$$\begin{aligned} \Pr(2\text{nd} = \text{white}) = {} & \Pr(2\text{nd} = \text{white} \mid 1\text{st} = \text{red}) \Pr(1\text{st} = \text{red}) \\ & + \Pr(2\text{nd} = \text{white} \mid 1\text{st} = \text{blue}) \Pr(1\text{st} = \text{blue}) \\ & + \Pr(2\text{nd} = \text{white} \mid 1\text{st} = \text{white}) \Pr(1\text{st} = \text{white}) \\ = {} & \frac{4}{11} . \frac{3}{12} + \frac{4}{11} . \frac{5}{12} + \frac{3}{11} . \frac{4}{12} = \frac{1}{3} \end{aligned}$$

Problem 2.11

(a) There are 2^n distinct words.
(b) Method 1:

$$Pr(2 \text{ ones}) = Pr(\{110\} \cup \{101\} \cup \{011\}) = Pr(110) + Pr(101) + Pr(011) = \frac{3}{8}.$$

Method 2:

$$Pr(2 \text{ ones}) = \binom{3}{2} \cdot \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{2}\right)^{3-2} = \frac{3}{8}.$$

Problem 2.16

$$\begin{aligned} Pr(\text{defective}) &= Pr(\text{defective} \mid A) \cdot Pr(A) + Pr(\text{defective} \mid B) \cdot Pr(B) \\ &= (0.15) \cdot \frac{1}{1.15} + (0.05) \cdot \frac{0.15}{1.15} = 0.137. \end{aligned}$$

Problem 2.26

(a) The probability mass function for a binomial random variable is given by

$$Pr(X = k) = \begin{pmatrix} n \\ k \end{pmatrix} p^k (1-p)^{n-k}$$

Tabulating the probabilities for various values of k we get the following

k	Probability
0	0.10737418240000
1	0.26843545600000
2	0.30198988800000
3	0.20132659200000
4	0.08808038400000
5	0.02642411520000
6	0.00550502400000
7	0.00078643200000
8	0.00007372800000
9	0.00000409600000
10	0.00000010240000

Refer to Figure 1 for the plot.

(b) The probability mass function for a Poisson distribution is given by

$$Pr(X = k) = e^{-\alpha} \frac{\alpha^k}{k!}$$

Tabulating the probabilities for various values of k we get the following

k	Probability
0	0.13533528323661
1	0.27067056647323
2	0.27067056647323
3	0.18044704431548
4	0.09022352215774
5	0.03608940886310
6	0.01202980295437
7	0.00343708655839
8	0.00085927163960
9	0.00019094925324
10	0.00003818985065

Refer to Figure 1 for the plot.

(c) Binomial: $Pr(X \geq 5) = 1 - Pr(X < 5) = 1 - \sum_{k=0}^{4} \binom{10}{k} \frac{1}{5}^k \frac{4}{5}^{10-k} = 0.0328$

Poisson: $Pr(X \geq 5) = 1 - Pr(X < 5) = 1 - \sum_{k=0}^{4} \frac{2^k}{k!} e^{-2} = 0.0527$

The Poisson approximation is not particularly good for this example.

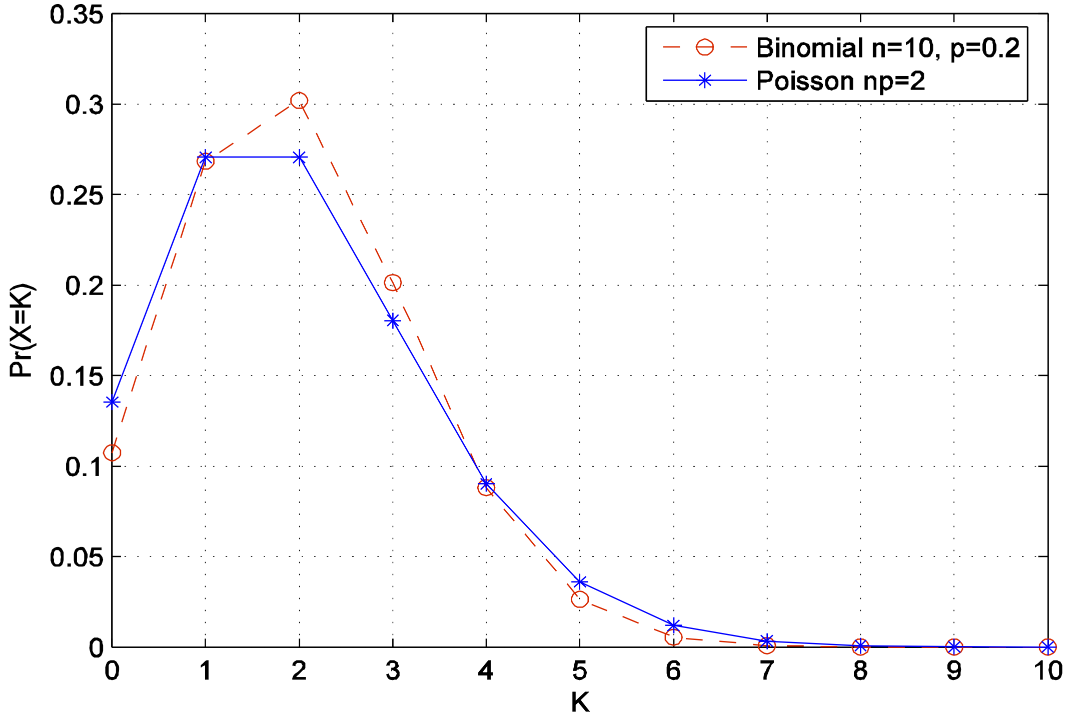

Figure 1 Probability Mass Function for Binomial and Poisson Distributions

Problem 2.29

Method 1:

$$\begin{aligned} Pr(X=0) &= \frac{4}{6}\cdot\frac{3}{5}\cdot\frac{2}{4}=\frac{1}{5}. \\ Pr(X=1) &= 3\cdot\frac{2}{6}\cdot\frac{4}{5}\cdot\frac{3}{4}=\frac{3}{5}. \\ Pr(X=2) &= 3\cdot\frac{4}{6}\cdot\frac{2}{5}\cdot\frac{1}{4}=\frac{1}{5}. \end{aligned}$$

Method 2:

$$P_X(k) = \frac{\binom{2}{k}\binom{4}{3-k}}{\binom{6}{3}} = \frac{\binom{2}{k}\binom{4}{3-k}}{20} \rightarrow \begin{cases} k=0 \rightarrow P_X(0) = \frac{1}{5} \\ k=1 \rightarrow P_X(1) = \frac{3}{5} \\ k=2 \rightarrow P_X(2) = \frac{1}{5} \end{cases}$$

Problem 2.30

Let $p = Pr(\text{success}) = \frac{1}{10}$.

$Pr(1 \text{ success}) = \binom{10}{1} \cdot p \cdot (1-p)^9 = 0.3874.$

$Pr(\geq 2 \text{ successes}) = 1 - Pr(\leq 1 \text{ success}) = 1 - Pr(0 \text{ successes}) - Pr(1 \text{ success}).$

$Pr(0 \text{ successes}) = (1-p)^1 0 = 0.3487.$

$Pr(\geq 2 \text{ successes}) = 1 - 0.3487 - 0.3874 = 0.2639.$