Solutions to Chapter 3 Exercises

Problem 3.3

$$f_X(x) = \begin{cases} a^{-bx} & x \le 0 \\ 0 & \text{otherwise} \end{cases}$$

a) Since this is a pdf the following integral should evaluate to 1

$$\int_{-\infty}^{+\infty} f_X(x)\,dx = 1$$

$$\int_{-\infty}^{0} f_X(x)\,dx + \int_{0}^{+\infty} f_X(x)\,dx = 1$$

$$\int_{-\infty}^{0} a^{-bx}dx + \int_{0}^{+\infty} 0dx = 1$$

$$\left.\frac{-a^{-bx}}{b\ln(a)}\right|_{-\infty}^{0} = 1$$

$$-\frac{1}{b\ln(a)} + \lim_{x\to-\infty}\frac{a^{-bx}}{b\ln(a)} = 1$$

Here two cases arise $a>1$ and $0<a<1$. The case $a<0$ is not possible because then $f_x(x)$ can become imaginary and will no longer be a valid pdf. The case $a=0$ is also not possible because then the denominator become zero. First considering the case $a>1$. This evaluates to a finite value only if $b<0$. And in the case $a<1$ the limit will be finite only if $b>0$. In both cases the limit evaluates to 0 and the above equation reduces to

$$-\frac{1}{b\ln(a)} = 1$$

$$a = e^{-1/b}$$

We can see that this relation satisfies the requirements we laid on a and b earlier. Also using this relation the pdf can be written as:

$$f_X(x) = \left(e^{-1/b}\right)^{-bx} = e^{x}$$

b) The CDF is given by

$$F_X(x) = \int_{-\infty}^{x} f_X(x)\,dx = \int_{-\infty}^{x} e^x dx = e^x$$

$$x = 0 \rightarrow F_X(0) = e^0 = 1 \rightarrow F_X(x) = \begin{cases} e^x & x \le 0 \\ 1 & x > 0 \end{cases}$$

Problem 3.4

a) Since

$$\int_{-\infty}^{+\infty} f_X(x)\;dx = 1$$

$$c\int_{-\infty}^{+\infty} e^{-2x}u(x)\;dx = 1$$

$$c\int_{0}^{+\infty} e^{-2x}\;dx = 1$$

$$-\frac{c}{2}e^{-2x}\Big|_{0}^{+\infty} = 1 \rightarrow c = 2,\ f_X(x) = 2e^{-2x}u(x)$$

b)

$$\Pr(X > 2) = 2\int_{2}^{+\infty} e^{-2x}\;dx = e^{-4} = 0.0183$$

c)

$$\Pr(X < 3) = 2\int_{0}^{3} e^{-2x}\;dx = 1 - e^{-6} = 0.9975$$

d)

$$\Pr(X < 3 | X > 2) = \frac{\Pr(2 < X < 3)}{\Pr(X > 2)}$$

$$= \frac{2\int_{2}^{3} e^{-2x}\;dx}{e^{-4}}$$

$$= \frac{e^{-4} - e^{-6}}{e^{-4}}$$

$$= 1 - e^{-2} = 0.8647$$

Problem 3.7

$$\Pr\left(\left|S-10\right|>0.075\right)=\int_{9.9}^{10-0.075} f_S\left(s\right)\,ds+\int_{10+0.075}^{10.1} f_S\left(s\right)\,ds$$

$$=2\int_{9.9}^{9.925} f_S\left(s\right)\,ds$$

$$=2\int_{9.9}^{9.925} 100\left(s-9.9\right)\,ds$$

$$s-9.9=u \quad \rightarrow \quad =2\int_{0}^{0.025} 100\,u\,du$$

$$=100u^2\Big|_0^{0.025}$$

$$=0.0625$$

Problem 3.10

$$f_X\left(x\right)=ce^{-2x^2-3x-1}$$

a) As usual the integral of the pdf evaluates to 1

$$2x^2+3x+1=2\left(x^2+\frac{3}{2}x+\frac{1}{2}\right)=2\left(x^2+\frac{3}{2}x+\frac{9}{16}-\frac{1}{16}\right)$$

$$=2\left(x^2+\frac{3}{2}x+\frac{9}{16}\right)-\frac{1}{8}=2\left(x+\frac{3}{4}\right)^2-\frac{1}{8}$$

$$\int_{-\infty}^{+\infty} f_X(x)\,dx=1\rightarrow\int_{-\infty}^{+\infty} ce^{-(2x^2+3x+1)}dx=1$$

$$\int_{-\infty}^{+\infty} ce^{-2\left(x+\frac{3}{4}\right)^2+\frac{1}{8}}dx=1$$

$$ce^{\frac{1}{8}}\int_{-\infty}^{+\infty} e^{-2\left(x+\frac{3}{4}\right)^2}dx=1$$

$$ce^{\frac{1}{8}}\sqrt{\frac{\pi}{2}}\int_{-\infty}^{+\infty}\frac{1}{\frac{1}{2}\sqrt{2\pi}}e^{\frac{-\left(x+\frac{3}{4}\right)^2}{2\left(\frac{1}{2}\right)^2}}dx=1$$

$$ce^{\frac{1}{8}}\sqrt{\frac{\pi}{2}}=1$$

$$c=e^{-\frac{1}{8}}\sqrt{\frac{2}{\pi}}=0.7041$$

b) The previous pdf can be rewritten in the form of a standard Gaussian pdf as follows

$$f_X(x)=ce^{-2x^2-3x-1}=e^{-\frac{1}{8}}\sqrt{\frac{2}{\pi}}\,e^{-2\left(x+\frac{3}{4}\right)^2+\frac{1}{8}}$$

$$=\sqrt{\frac{2}{\pi}}\,e^{-2\left(x+\frac{3}{4}\right)^2}$$

This is in the form of standard Gaussian pdf given by $f_X(x)=\sqrt{\frac{1}{2\pi\sigma^2}}\exp\left\{\frac{-(x-m)^2}{2\sigma^2}\right\}$ and we can easily identify the mean m and the standard deviation σ as

$$m=-\frac{3}{4},\quad \sigma^2=\frac{1}{4}\rightarrow\sigma=\frac{1}{2}$$

Problem 3.16

$$\Pr(M=0)=P_0$$
$$\Pr(M=0)=P_1$$
$$f_{X|M=0}(x)=\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{x^2}{2\sigma^2}\right\}$$
$$f_{X|M=1}(x)=\frac{1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{(x-1)^2}{2\sigma^2}\right\}$$
$$\begin{aligned} f_X(x) &= f_{X|M=0}(x)\Pr(M=0)+f_{X|M=1}(x)\Pr(M=1) \\ &= f_{X|M=0}(x)\,P_0+f_{X|M=1}(x)\,P_1 \end{aligned}$$
$$\begin{aligned} \Pr(M=0|X=x) &= \frac{f_{X|M=0}(x)\Pr(M=0)}{f_X(x)} \\ &= \frac{f_{X|M=0}(x)\,P_0}{f_X(x)} \\ &= \frac{\frac{P_0}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{x^2}{2\sigma^2}\right\}}{\frac{P_0}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{x^2}{2\sigma^2}\right\}+\frac{P_1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{(x-1)^2}{2\sigma^2}\right\}} \\ &= \frac{1}{1+\frac{P_1}{P_0}\exp\left\{\frac{2x-1}{2\sigma^2}\right\}} \end{aligned}$$

a)

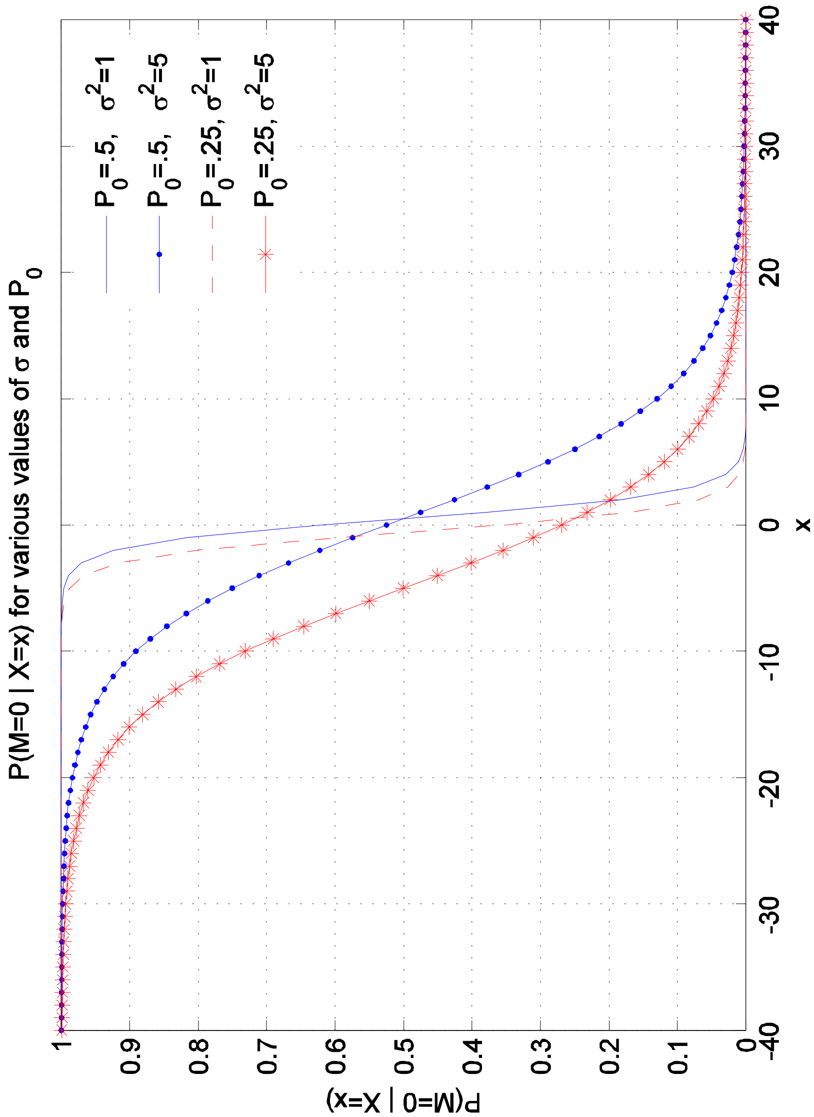
$$P_0=P_1=0.5,\ \sigma^2=1$$
$$\rightarrow \Pr(M=0|X=x)=\frac{1}{1+\exp\{x-0.5\}}$$

$$P_0=P_1=0.5,\ \sigma^2=5$$
$$\rightarrow \Pr(M=0|X=x)=\frac{1}{1+\exp\left\{\frac{2x-1}{10}\right\}}$$

b)

$$P_0=0.25,\ P_1=0.75,\ \sigma^2=1$$
$$\rightarrow \Pr(M=0|X=x)=\frac{1}{1+3\exp\{x-0.5\}}$$

$$P_0=0.25,\ P_1=0.75,\ \sigma^2=5$$
$$\rightarrow \Pr(M=0|X=x)=\frac{1}{1+3\exp\left\{\frac{2x-1}{10}\right\}}$$



P(M=0 | X=x) for various values of σ and P0
P0=.5, σ2=1
P0=.5, σ2=5
P0=.25, σ2=1
P0=.25, σ2=5
P(M=0 | X=x)
x

Figure 1

Problem 3.17

a) We decide 0 if $\Pr\left(M=0\middle|X=x\right)\ge 0.9$. Using the achieved result in problem 3.16

$$\text{for} \qquad P_0 = P_1 = 0.5,\ \sigma^2 = 1$$

$$\Pr\left(M=0\middle|X=x\right)=\frac{1}{1+\exp\{x-0.5\}}\ge 0.9$$

$$\rightarrow \exp\{x-0.5\}\le\frac{1}{9}$$

$$\rightarrow x\le -1.6972$$

In the same way, we decide 1 if $\Pr\left(M=1\middle|X=x\right)\ge 0.9$.

$$\Pr(M=1|X=x)=\frac{f_{X|M=1}(x)\Pr(M=1)}{f_X(x)}$$

$$=\frac{f_{X|M=1}(x)\,P_1}{f_X(x)}$$

$$=\frac{\frac{P_1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{(x-1)^2}{2\sigma^2}\right\}}{\frac{P_0}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{x^2}{2\sigma^2}\right\}+\frac{P_1}{\sqrt{2\pi\sigma^2}}\exp\left\{-\frac{(x-1)^2}{2\sigma^2}\right\}}$$

$$=\frac{1}{1+\frac{P_0}{P_1}\exp\left\{\frac{1-2x}{2\sigma^2}\right\}}$$

$$\text{for} \qquad P_0 = P_1 = 0.5,\ \sigma^2 = 1$$

$$\Pr\left(M=1\middle|X=x\right)=\frac{1}{1+\exp\{0.5-x\}}\ge 0.9$$

$$\rightarrow \exp\{0.5-x\}\le\frac{1}{9}$$

$$\rightarrow x\ge 2.6972$$

In the end we can formulize the decision rules as follows

$$\begin{cases} 0 & x \le -1.6972 \\ 1 & x \ge 2.6972 \\ Erased & -1.6972 < x < 2.6972 \end{cases}$$

b)

$$\begin{aligned} P(Erased) &= P(-1.6972 < x < 2.6972) \\ &= P(-1.6972 < x < 2.6972 | M=0)\, P_0 + P(-1.6972 < x < 2.6972 | M=1)\, P_1 \\ &= \frac{P(-1.6972 < x < 2.6972 | M=0)}{2} + \frac{P(-1.6972 < x < 2.6972 | M=1)}{2} \end{aligned}$$

$$\begin{cases} x|M=0:\ N(0,1) \\ x|M=1:\ N(1,1) \\ F_X(x) = 1 - Q\left(\dfrac{x-m}{\sigma}\right) \end{cases} \rightarrow P(Erased) = \frac{Q(-1.6972) - Q(2.6972)}{2} + \frac{Q(-1.6972-1) - Q(2.6972-1)}{2}$$

$$\begin{aligned} &= \frac{1 - Q(1.6972) - Q(2.6972)}{2} + \frac{1 - Q(2.6972) - Q(1.6972)}{2} \\ &= 1 - Q(1.6972) - Q(2.6972) \\ &= 0.95196 \end{aligned}$$

c)

$$\begin{aligned} P(Error) &= P(Error | M=0)\, P_0 + P(Error | M=1)\, P_1 \\ &= \frac{P(x \ge 2.6972 | M=0)}{2} + \frac{P(x \le -1.6972 | M=1)}{2} \\ &= \frac{Q(2.6972)}{2} + \frac{1 - Q(-1.6972-1)}{2} \\ &= Q(2.6972) \\ &= 0.00347 \end{aligned}$$