Solutions to Chapter 10 and 11 Exercises

Problem 10.8

Let

$$s(t) = \sum_{k=-\infty}^{\infty} s_k \exp(j2\pi f_o t).$$

Then

$$\begin{aligned}
s(t-T) &= \sum_{k=-\infty}^{\infty} s_k \exp(j2\pi f_o(t-T)) \\
R_{X,X}(\tau) &= E\left[\sum_k \sum_m s_k s_m^* \exp(j2\pi k f_o(t-T)\exp(-j2\pi m f_o(t+\tau-T)\right] \\
&= \sum_k \sum_m s_k s_m^* \exp(j2\pi(k-m)f_o t)\exp(-j2\pi m f_o \tau) \\
&\quad E[\exp(j2\pi(k-m)f_o T)] \\
E[\exp(j2\pi(k-m)f_o T)] &= \frac{1}{t_o}\int_0^{t_o} \exp(j2\pi(k-m)f_o u)du \\
&= \begin{cases} 0 & k \neq m \\ 1 & k = m \end{cases} \\
R_{X,X}(\tau) &= \sum_{k=-\infty}^{\infty} |s_k|^2 \exp(-j2\pi k f_o \tau) \\
S_{X,X}(f) &= \sum_{k=-\infty}^{\infty} |s_k|^2 \delta(f - k f_o)
\end{aligned}$$

Hence the process X(t)=s(t-T) has a line spectrum and height of each line is given by the magnitude squared of the Fourier series coefficients.

Problem 10.12

$$R_{X,X}(t_1,t_2) = E[\cos(\omega_c t_1 + B[n_1]\pi/2)\cos(\omega_c t_2 + B[n_2]\pi/2)],$$

where n_1 and n_2 are integers such that $n_1 T \leq t_1 < (n_1+1)T$ and $n_2 T \leq t_2 < (n_2+1)T/$. For t_1, t_2 such that $n_1 \neq n_2$,

$$R_{X,X}(t_1,t_2) = E[\cos(\omega_c t_1 + B[n_1]\pi/2)]E[\cos(\omega_c t_2 + B[n_2]\pi/2)] = 0,$$

while for t_1, t_2 such that $n_1 = n_2$,

$$\begin{aligned} R_{X,X}(t_1,t_2) &= \frac{1}{2}\cos(\omega_c(t_2-t_1)) + \frac{1}{2}E[\cos(\omega_c(t_2+t_1)+\pi B[n_1])] \\ &= \frac{1}{2}\cos(\omega_c(t_2-t_1)) - \frac{1}{2}\cos(\omega_c(t_2+t_1)) \end{aligned}$$

Since this autocorrelation depends on more than just $t_1 - t_2$, the process is not WSS.
(b) From part (a),

$$R_{X,X}(t,t+\tau) = \begin{cases} 0 & \text{if } t, t+\tau \text{ are in different intervals,} \\ \frac{1}{2}\cos(\omega_c\tau) - \frac{1}{2}\cos(\omega_c(2t+\tau)) & \text{if } t, t+\tau \text{ are in in the same intervals.} \end{cases}$$

Since the process is not WSS we must take time averages.

$$R_{X,X}(\tau) = \langle R_{X,X}(t,t+\tau)\rangle = (1-p(\tau))\langle 0\rangle + p(\tau)\langle \frac{1}{2}\cos(\omega_c\tau) - \frac{1}{2}\cos(\omega_c(2t+\tau))\rangle,$$

where $p(\tau)$ is the fraction of the values of t that lead to t and $t+\tau$ being in the same interval. This function is given by

$$p(\tau) = \begin{cases} 0 & |\tau| > T, \\ 1 - \frac{|\tau|}{T} & |\tau| < T. \end{cases}$$

Therefore,

$$\begin{aligned} p(\tau) &= \text{tri}(t/T) \\ R_{X,X}(\tau) &= \frac{1}{2}\text{tri}(\tau/T)\cos(\omega_c\tau) \\ S_{X,X}(f) &= \frac{1}{2}FT[\text{tri}(\tau/T)] * FT[\cos(\omega_c\tau] \end{aligned}$$

using Table E.1 in Appendix E in the text,

$$\begin{aligned} S_{X,X}(f) &= \frac{1}{4}T\text{sinc}^2(fT) * (\delta(f-f_c) + \delta(f+f_c)) \\ &= \frac{T}{4}(\text{sinc}^2((f-f_c)T)) + \text{sinc}^2((f+f_c)T)) \end{aligned}$$

Problem 10.14

(a) The absolute BW is ∞ since $S(f) > 0$ for all $|f| < \infty$.
(b) The 3dB BW, f_3 satisfies

$$\begin{aligned}
\frac{1}{(1+(f_3/B)^2)^3} &= \frac{1}{2} \\
\Rightarrow f_3 &= B\sqrt{2^{1/3}-1} = 0.5098B.
\end{aligned}$$

(c)

$$\begin{aligned}
\int_{-\infty}^{\infty} f^2 S(f)df &= \int_{-\infty}^{\infty} \frac{f^2}{(1+(f/B)^2)^3} df = B^3 \int_{-\infty}^{\infty} \frac{z^2}{(1+z^2)^3} dz = \frac{\pi}{8} B^3 \\
\int_{-\infty}^{\infty} S(f)df &= \int_{-\infty}^{\infty} \frac{1}{(1+(f/B)^2)^3} df = B \int_{-\infty}^{\infty} \frac{1}{(1+z^2)^3} dz = \frac{3\pi}{8} B \\
B_{rms}^2 &= \frac{\frac{\pi}{8}B^3}{\frac{3\pi}{8}B} = \frac{B^2}{3} \\
B_{rms} &= \frac{B}{\sqrt{3}}.
\end{aligned}$$

Problem 10.19

(a)

$$\begin{aligned} E[\epsilon^2] &= E[(Y[n+1] - a_1 Y[n] - a_2 Y[n-1])^2] \\ &= R_{Y,Y}[0](1 + a_1^2 + a_2^2) - 2a_1(1-a_2)R_{Y,Y}[1] - 2a_2 R_{Y,Y}[2] \end{aligned}$$

(b)

$$\begin{aligned} \frac{\partial E[\epsilon^2]}{\partial a_1} &= 2a_1 R_{Y,Y}[0] - 2(1-a_2)R_{Y,Y}[1] = 0 \\ &\Rightarrow R_{Y,Y}[0]a_1 + R_{Y,Y}[1]a_2 = R_{Y,Y}[1] \\ \frac{\partial E[\epsilon^2]}{\partial a_2} &= 2a_2 R_{Y,Y}[0] - 2R_{Y,Y}[2] + 2a_1 R_{Y,Y}[1] = 0 \\ &\Rightarrow R_{Y,Y}[1]a_1 + R_{Y,Y}[0]a_2 = R_{Y,Y}[2] \\ &\Rightarrow \begin{bmatrix} R_{Y,Y}[0] R_{Y,Y}[1] & \\ R_{Y,Y}[1] & R_{Y,Y}[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} R_{Y,Y}[1] \\ R_{Y,Y}[2] \end{bmatrix} \\ \Rightarrow \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} &= \frac{1}{R_{Y,Y}^2[0] - R_{Y,Y}^2[1]} \begin{bmatrix} R_{Y,Y}[0]R_{Y,Y}[1] - R_{Y,Y}[1]R_{Y,Y}[2] \\ R_{Y,Y}[0]R_{Y,Y}[2] - R_{Y,Y}^2[1] \end{bmatrix} \end{aligned}$$

Problem 11.10

$$
\begin{aligned}
Y(t) &= a(S^2(t) + N^2(t) + 2S(t)N(t)) \\
E[Y(t)] &= aE[S^2(t)] + aE[N^2(t)] + 2aE[S(t)N(t)] \\
&= a(\sigma_S^2 + \sigma_N^2). \\
R_{Y,Y}(\tau) &= a^2E[(S^2(t) + N^2(t) + 2S(t)N(t)) \\
&\quad (S^2(t+\tau) + N^2(t+\tau) + 2S(t+\tau)N(t+\tau))] \\
&= a^2E[S^2(t)S^2(t+\tau)] + a^2E[N^2(t)N^2(t+\tau)] \\
&+ 4a^2E[S(t)S(t+\tau)N(t)N(t+\tau)] + a^2E[S^2(t)N^2(t+\tau)] \\
&+ 2a^2E[S^2(t)S(t+\tau)N(t+\tau)] + a^2E[N^2(t)S^2(t+\tau)] \\
&+ 2a^2E[N^2(t)N(t+\tau)S(t+\tau)] + 2a^2E[S(t)S^2(t+\tau)N(t)] \\
&+ 2a^2E[S(t)N(t)N^2(t+\tau)] \\
&= a^2[R_{S,S}^2(0) + 2R_{S,S}^2(\tau) + R_{N,N}^2(0) + 2R_{N,N}^2(\tau) \\
&+ 4R_{S,S}(\tau)R_{N,N}(\tau) + R_{S,S}(0)R_{N,N}(0) + R_{N,N}(0)R_{S,S}(0)] \\
&= a^2[(R_{S,S}(0) + R_{N,N}(0))^2 + 2(R_{S,S}(\tau) + R_{N,N}(\tau))^2] \\
&= a^2[(\sigma_S^2 + \sigma_N^2)^2 + 2(R_{S,S}(\tau) + R_{N,N}(\tau))^2].
\end{aligned}
$$

In above Calculation we used the following information:

For two Gaussian random process X(t) and Y(t) :

$E\left[X^2(t)\,Y^2(t+\tau)\right] = R_{XX}[0]\,R_{YY}[0] + 2R_{XY}^2[\tau]$ If X and Y are correlated

$E\left[X(t)\,Y(t)\,Y(t+\tau)\right] = E\left[X(t)\right]E\left[Y(t)\,Y(t+\tau)\right]$ If X and Y are not correlated

Problem 11.11

For the given PSD's,

$$\begin{aligned}
R_{S,S}(\tau) &= F^{-1}[S_{S,S}(f)] = \frac{A^2}{4}(e^{-j2\pi f_c\tau} + e^{j2\pi f_c\tau}) = \frac{A^2}{2}\cos(2\pi f_c\tau) \\
R_{N,N}(\tau) &= F^{-1}[S_{N,N}(f)] = \frac{N_oB}{2}\text{sinc}(B\tau)(e^{-j2\pi f_c\tau} + e^{j2\pi f_c\tau}) \\
&= N_oB\text{sinc}(B\tau)\cos(2\pi f_c\tau)
\end{aligned}$$

From these we determine that

$$\begin{aligned}
\sigma_S^2 &= R_{S,S}(0) = \frac{A^2}{2}, \\
\sigma_N^2 &= R_{N,N}(0) = N_oB.
\end{aligned}$$

The autocorrelation of $Y(t)$ is then

$$\begin{aligned}
R_{Y,Y}(\tau) &= a^2[(\sigma_S^2 + \sigma_N^2)^2 + 2(R_{S,S}(\tau) + R_{N,N}(\tau))^2] \\
&= a^2\left[\left(\frac{A^2}{2} + N_oB\right)^2 + 2\left(\frac{A^2}{2} + N_oB\text{sinc}(B\tau)\right)^2\cos^2(2\pi f_c\tau)\right] \\
&= a^2\Bigg[\left(\frac{A^2}{2} + N_oB\right)^2 \\
&+ 2\left(\frac{A^2}{4} + A^2N_oB\text{sinc}(B\tau) + (N_oB)^2\text{sinc}^2(B\tau)\right)(1 + \cos(4\pi f_c\tau))\Bigg]
\end{aligned}$$

Taking FT's, the PSD of $Y(t)$ is then

$$\begin{aligned}
S_{Y,Y}(f) &= a^2\Bigg[\left(\frac{A^2}{2} + N_oB\right)^2\delta(f) + 2\left(\frac{A^2}{4}\delta(f) + A^2N_o\text{rect}(f/B) + N_o^2B\text{tri}(f/B)\right) \\
&\quad * \left(\delta(f) + \frac{1}{2}\delta(f-2f_c) + \frac{1}{2}\delta(f+2f_c)\right)\Bigg] \\
&= a^2\left(\frac{A^2}{2} + A^2N_oB + (N_oB)^2\right)\delta(f) + A^2N_o\text{rect}(f/B) + N_o^2B\text{tri}(f/B) \\
&+ \frac{A^2}{4}(\delta(f-2f_c) + \delta(f+2f_c)) + A^2N_o\left(\text{rect}\left(\frac{f-2f_c}{B}\right) + \text{rect}\left(\frac{f+2f_c}{B}\right)\right) \\
&+ N_o^2B\left(\text{tri}\left(\frac{f-2f_c}{B}\right) + \text{tri}\left(\frac{f+2f_c}{B}\right)\right).
\end{aligned}$$

Problem 11.14

$$S_{Y,Y}(f) = S_{X,X}(f)|H(f)|^2 = \frac{a}{1+(f/f_o)^2} \cdot b^2 \left[\mathrm{rect}\left(\frac{f-f_c}{f_2-f_1}\right) + \mathrm{rect}\left(\frac{f+f_c}{f_2-f_1}\right)\right],$$

where $f_c = (f_1 + f_2)/2$. Assuming $f_2 - f_1 << f_o$, the input PSD will be approximately constant over the passband of the filter. In that case, the output PSD will be well approximated by

$$\begin{aligned} S_{Y,Y}(f) &\approx S_{X,X}(f_c)|H(f)|^2 = \frac{ab^2}{1+(f_c/f_o)^2} \cdot \left[\mathrm{rect}\left(\frac{f-f_c}{f_2-f_1}\right) + \mathrm{rect}\left(\frac{f+f_c}{f_2-f_1}\right)\right] \\ \Rightarrow &= \frac{2ab^2(f_2-f_1)}{1+(f_c/f_o)^2} \mathrm{sinc}((f_2-f_1)\tau)\cos(\omega_c \tau). \end{aligned}$$

Problem 11.26

$h(t)=s(t_0-t)$. In this case, $s(t_0-t)=s(t)$ so the impulse response of the matched filter is the same as the signal itself.