Solutions to Chapter 7 Exercises

Problem 7.3

Since the N_i are Gaussian, $\hat{\mu}$ is also Gaussian with

$$\begin{aligned} E[\hat{\mu}] &= \mu_N = 0 \\ Var(\hat{\mu}) &= \frac{\sigma_N^2}{n} = \frac{0.01}{100} = 10^{-4}. \\ \Rightarrow \hat{\mu} &\sim N(0, 10^{-4}). \end{aligned}$$

Problem 7.8

Given $x_1, x_2, \ldots, x_N$ are observed, we want to minimize

$$\epsilon^2 = \frac{1}{N}\sum_{n=1}^{N}(x_n - a - bn)^2.$$

Taking derivatives with respect to a and b and setting equal to zero produces

$$\begin{aligned} \frac{\partial \epsilon^2}{\partial a} &= \frac{1}{N}\sum_{n=1}^{N}(-2)(x_n - a - bn) = 0 \\ \Rightarrow \frac{1}{N}\sum_{n=1}^{N} x_n &= a\left(\frac{1}{N}\sum_{n=1}^{N} 1\right) + b\left(\frac{1}{N}\sum_{n=1}^{N} n\right) \\ \frac{\partial \epsilon^2}{\partial b} &= \frac{1}{N}\sum_{n=1}^{N}(-2n)(x_n - a - bn) = 0 \\ \Rightarrow \frac{1}{N}\sum_{n=1}^{N} n x_n &= a\left(\frac{1}{N}\sum_{n=1}^{N} n\right) + b\left(\frac{1}{N}\sum_{n=1}^{N} n^2\right) \end{aligned}$$

To simplify the notation, define the following:

$$\begin{aligned}\overline{n} &= \frac{1}{N}\sum_{n=1}^{N} n,\\ \overline{n^2} &= \frac{1}{N}\sum_{n=1}^{N} n^2,\\ \overline{x_n} &= \frac{1}{N}\sum_{n=1}^{N} x_n,\\ \overline{nx_n} &= \frac{1}{N}\sum_{n=1}^{N} nx_n.\end{aligned}$$

Then, the optimum values of a and b will satisfy the following matrix equation:

$$\begin{bmatrix}1 & \overline{n}\\ \overline{n} & \overline{n^2}\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix} = \begin{bmatrix}\overline{x_n}\\ \overline{nx_n}\end{bmatrix}$$

. The solution is

$$\begin{bmatrix}a\\ b\end{bmatrix} = \frac{\begin{bmatrix}\overline{n^2}\cdot\overline{x_n} - \overline{n}\cdot\overline{nx_n}\\ \overline{nx_n} - \overline{n}\cdot\overline{x_n}\end{bmatrix}}{\overline{n^2} - (\overline{n})^2}.$$

Problem 7.11

(a) Because X_i is a Bernoulli RV

$$\begin{aligned}\sigma_X^2 &= p_A(1-p_A)\\ \hat{p_A} &= \frac{1}{n}\sum_{i=1}^{n} X_i\\ E[\hat{p_A}] &= p_A\\ Var(\hat{p_A}) &= \frac{\sigma_X^2}{n} = \frac{p_A(1-p_A)}{n}\end{aligned}$$

By virtue of the central limit theorem, we can write

$$\hat{p_A} \sim (p_A, \frac{p_A(1-p_A)}{n})$$

$$\begin{aligned}Pr(|\hat{p_A} - p_A| < \varepsilon) &= 1 - 2Q\left(\frac{p_A + \varepsilon - p_A}{\sqrt{p_A(1-p_A)/n}}\right)\\ &= 1 - 2Q\left(\sqrt{\frac{n\varepsilon^2}{p_A(1-p_A)}}\right)\end{aligned}$$

(b)

$$Pr(|\hat{p_A} - p_A| < 0.1p_A) \;=\; 0.95$$

Using the result from (a) we get

$$\begin{aligned}
1 - 2Q\left(\sqrt{\frac{n(0.1p_A)^2}{p_A(1-p_A)}}\right) &= 0.95 \\
\Rightarrow Q\left(\sqrt{\frac{0.01p_A n}{(1-p_A)}}\right) &\leq 0.025 \\
\Rightarrow \sqrt{\frac{0.01p_A n}{(1-p_A)}} &\geq 1.9597 \approx 1.96
\end{aligned}$$

Note in the last step, the inequality is reversed since $Q(x)$ is a decreasing function of x.

$$\Rightarrow n \geq 19.6^2 \frac{1-p_A}{p_A}$$

(c)

$$\begin{aligned}
Y_n &= \sum_{i=1}^{n} X_i = n\hat{p_A} \\
E[Y_n] &= np_A
\end{aligned}$$

Since the value of n was chosen to satisfy the constraints of (b), we can write

$$E[Y_n] \;=\; 19.6^2 \frac{1-p_A}{p_A} p_A = 19.6^2(1-p_A).$$

Strictly speaking we will have

$$E[Y_n] \;\geq\; 19.6^2(1-p_A).$$

If we assume that $p_A << 1$ we can approximate it as

$$E[Y_n] \;\geq\; 19.6^2 \approx 384.$$

Problem 7.14

$$\mu_X = 5\text{volts}, \sigma_X = 0.25\text{volts}.$$

For $n = 100$ samples, the sample mean will have

$$E[\hat{\mu}] = 5\text{volts}, \sigma_{\hat{\mu}} = \frac{1}{40}\text{volts}.$$

The 99% confidence interval will be $(\mu_X - \epsilon, \mu_X + \epsilon)$ where

$$\epsilon = c_{0.99}\sigma_{\hat{\mu}} = 2.58 \cdot \frac{1}{40} = 0.0645\text{volts}.$$

Hence, the 99% confidence interval is (4.9355,5.0645) volts. None of the estimates in (a)-(c) fall in this range.