Solutions to Chapter 5 Exercises (Part 2)

Problem 5.5

$$f_{X,Y}(x,y) = \frac{1}{8\pi} \exp\left(-\frac{(x-1)^2 + (y+1)^2}{8}\right)$$

(a) $Pr(X > 2, Y < 0)$

$$
\begin{aligned}
Pr(X > 2, Y < 0) &= \int_2^\infty \int_{-\infty}^0 \frac{1}{8\pi} \exp\left(-\frac{(x-1)^2 + (y+1)^2}{8}\right) \\
&= \int_2^\infty \frac{1}{\sqrt{8\pi}} \exp\left(-\frac{(x-1)^2}{8}\right) dx \int_{-\infty}^0 \frac{1}{\sqrt{8\pi}} \exp\left(-\frac{(y+1)^2}{8}\right) dy \\
&= Q\left(\frac{2-1}{2}\right) \Phi\left(\frac{0+1}{2}\right) = Q\left(\frac{1}{2}\right) \Phi\left(\frac{1}{2}\right) \\
&= Q(1/2)\left(1 - Q(1/2)\right)
\end{aligned}
$$

(b) $Pr(0 < X < 2, |Y+1| > 2)$

$$
\begin{aligned}
&= Pr(\{0 < X < 2, Y > 1\} \cup \{0 < X < 2, Y < -3\}) \\
&= \int_0^2 \int_{-\infty}^{-3} f_{X,Y}(x,y) dx\, dy + \int_0^2 \int_1^\infty f_{X,Y}(x,y) dx\, dy \\
&= \int_0^2 \frac{1}{\sqrt{8\pi}} exp\left(-\frac{(x-1)^2}{8}\right) dx \left(\int_{-\infty}^{-3} \frac{1}{\sqrt{8\pi}} \exp\left(-\frac{(y+1)^2}{8}\right) dy + \int_1^\infty \frac{1}{\sqrt{8\pi}} \exp\left(-\frac{(y+1)^2}{8}\right) dy\right) \\
&= \left(Q\left(\frac{0-1}{2}\right) - Q\left(\frac{2-1}{2}\right)\right)\left(\Phi\left(\frac{-3+1}{2}\right) + Q\left(\frac{1+1}{2}\right)\right) \\
&= \left(Q\left(-\frac{1}{2}\right) - Q\left(\frac{1}{2}\right)\right)\left(\Phi(-1) + Q(1)\right) \\
&= 2Q(1)\left(1 - 2Q\left(\frac{1}{2}\right)\right)
\end{aligned}
$$

(c) $Pr(Y > X)$

$$Pr(Y > X) = \int_{-\infty}^{\infty}\int_{-\infty}^{y} f_{X,Y}(x,y)dx\,dy$$

This integral is easier to do with a change of variables. Let us use the substitution

$$u = x - y$$
$$v = x + y$$
$$J\begin{pmatrix} u & v \\ x & y \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$
$$\left|det\begin{bmatrix} u & v \\ x & y \end{bmatrix}\right| = 2$$
$$f_{U,V}(u,v) = \frac{f_{X,Y}(x,y)}{2}$$
$$Pr(Y > X) = Pr(U < 0)$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{0} \frac{1}{8\pi} \exp\left(-\frac{(x-1)^2 + (y+1)^2}{8}\right)\frac{1}{2}\,du\,dv$$
$$= \frac{1}{16\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{0} \exp\left(-\frac{(u-v-2)^2 + (u+v+2)^2}{32}\right)\,du\,dv$$
$$= \frac{1}{16\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{0} \exp\left(-\frac{(u^2 + v^2 + 4 - 4u)}{16}\right)\,du\,dv$$

$$= \frac{1}{16\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{0} \exp\left(-\frac{(u-2)^2 + v^2}{16}\right)\,du\,dv$$
$$= \frac{1}{16\pi}\int_{-\infty}^{\infty} \exp\left(-\frac{v^2}{16}\right)\,dv\int_{-\infty}^{0} \exp\left(-\frac{(u-2)^2}{16}\right)\,du$$
$$= \frac{1}{16\pi}\sqrt{16\pi}\sqrt{16\pi}\,\Phi\left(\frac{0-2}{\sqrt{8}}\right)$$
$$= Q(\frac{1}{\sqrt{2}})$$

Problem 5.29

Note that any linear transformation of jointly Gaussian random variables produces jointly Gaussian random variables. So U, V are both Gaussian random variables. And if $\rho_{U,V} = 0$, U and V are independent.

$$E[U] = aE[X] + bE[Y] = 0 .$$

$$E[V] = cE[X] + dE[Y] = 0 .$$

$$E[UV] = acE[X^2] + bdE[Y^2] + (ad + bc)E[XY] = ac + bd + \rho(ad + bc) .$$

Thus, $\rho_{U,V} = 0$ if

$$ac + bd + \rho(ad + bc) = 0 .$$

So, U and V are independent, if

$$ac + bd = -\rho(ad + bc) .$$

Problem 5.32

$$Z = X^2 + Y^2$$
$$W = XY$$
$$X = \frac{\sqrt{Z+2W} + \sqrt{Z-2W}}{2}$$
$$Y = \frac{\sqrt{Z+2W} - \sqrt{Z-2W}}{2}$$
$$f_X(x) = \frac{1}{\pi(1+x^2)}$$
$$f_Y(y) = \frac{1}{\pi(1+y^2)}$$
$$J\begin{pmatrix} z & w \\ x & y \end{pmatrix} = \begin{pmatrix} 2x & 2y \\ y & x \end{pmatrix}$$
$$\left| det\left[J\begin{pmatrix} z & w \\ x & y \end{pmatrix} \right] \right| = 2|x^2 - y^2|$$
$$f_{Z,W}(z,w) = \frac{f_X(x)f_Y(y)}{2|x^2 - y^2|}$$
$$= \frac{1}{2\pi^2}\frac{1}{(1+x^2)(1+y^2)|x^2 - y^2|}$$
$$= \frac{1}{2\pi^2}\frac{1}{(1+x^2+y^2+x^2y^2)|x^2 - y^2|}$$
$$= \frac{1}{2\pi^2}\frac{1}{(1+z^2+w^2)\sqrt{z^2 - 4w^2}}$$