Solutions to Chapter 6 Exercises (Part 2)

Problem 6.3

(a) Define

$$I_n(z) = \int\int \dots \int_{\sum_{i=1}^{n} x_i \le z} dx_1 dx_2 \dots dx_n.$$

Then $c = I_N^{-1}(1)$. Note that

$$\begin{aligned} I_n(z) &= \int_0^z \left[\int\int \dots \int_{\sum_{i=1}^{n-1} x_i \le z - x_n} dx_1 dx_2 \dots dx_{n-1} \right] dx_n \\ &= \int_0^z I_{n-1}(z - x_n) dx_n \\ &= \int_0^z I_{n-1}(u) du. \end{aligned}$$

Using this iteration, we see that

$$I_1(z) = \int_0^z du = z,$$

$$
\begin{aligned}
I_2(z) &= \int_0^z u du = \frac{1}{2}z^2, \\
I_3(z) &= \int_0^z \frac{1}{2}u^2 du = \frac{1}{3!}z^3, \\
&\vdots \\
I_n(z) &= \frac{1}{n!}z^n.
\end{aligned}
$$

Then, $c = I_N^{-1}(1) = N!$.

(b)

$$
\begin{aligned}
f_{X_1,X_2,...,X_M}(x_1,x_2,...,x_M) &= \int\int\cdots\int_{x_{M+1}+\ldots+x_N \le 1-x_1-x_2-\ldots-x_M} c\,dx_N dx_{N-1}\ldots dx_{M+1} \\
&= c\,I_{N-M}(1-x_1-x_2-\ldots-x_M) \\
&= \frac{N!}{(N-M)!}(1-x_1-\ldots-x_M)^{N-M}\ , \qquad \sum_{m=1}^{M} x_M \le 1, x_i \ge 0
\end{aligned}
$$

(c) From part(b), we have

$$f_{X_i}(x_i) = N(1-x_i)^{N-1}\ .$$

Then,

$$f_{X_1}(x_1)f_{X_1}(x_1)...f_{X_N}(x_N) = N^N(1-x_1)^{N-1}...(1-x_N)^{N-1} \neq f_{X_1,X_2,...,X_N}(x_1,x_2,...,x_N) = N!$$

Thus, X_i are identically distributed, but they are not independent.

Problem 6.12

$$
\begin{aligned}
\mathbf{X} &= [X_1, X_2, X_3, X_4]^T \\
\Omega &= [\omega_1, \omega_2, \omega_3, \omega_4]^T
\end{aligned}
$$

From problem 6.11 we know

$$
\begin{aligned}
&\mathbf{X} = [X_1, X_2, X_3, X_4]^T \\
&\Phi_{\mathbf{X}}(\Omega) = \exp\left(-\frac{\Omega^T \mathbf{C}_{\mathbf{XX}} \Omega}{2}\right) \\
&\rightarrow E\left[e^{j(\omega_1 X_1 + \omega_2 X_2 + \omega_3 X_3 + \omega_4 X_4)}\right] = \exp\left(-\frac{\Omega^T \mathbf{C}_{\mathbf{XX}} \Omega}{2}\right)
\end{aligned}
$$

Expanding both the exponentials and equating the coefficients of $\omega_1\omega_2\omega_3\omega_4$ we get

$$E\left[1+j\left(\omega_1 X_1+\omega_2 X_2+\omega_3 X_3+\omega_4 X_4\right)-\frac{1}{2!}\left(\omega_1 X_1+\omega_2 X_2+\omega_3 X_3+\omega_4 X_4\right)^2+\ldots\right]$$
$$=\exp\left(1-\frac{\Omega^T \mathbf{C}_{\mathbf{XX}}\Omega}{2}+\frac{1}{2!}\left(\frac{\Omega^T \mathbf{C}_{\mathbf{XX}}\Omega}{2}\right)^2+\ldots\right)$$
$$\rightarrow E\left[\ldots+X_1 X_2 X_3 X_4 \omega_1\omega_2\omega_3\omega_4+\ldots\right]=\left(\ldots+\frac{1}{2!}\frac{1}{2^2}\left(c_{12}c_{34}+c_{13}c_{24}+c_{14}c_{23}\right)8\omega_1\omega_2\omega_3\omega_4+\ldots\right)$$

Where $c_{ij}=\operatorname{cov}\left(X_i,X_j\right)$

$$\rightarrow E\left[X_1X_2X_3X_4\right]=\left(\frac{1}{2!}\frac{1}{2^2}\left(c_{12}c_{34}+c_{13}c_{24}+c_{14}c_{23}\right)8\right)$$
$$\rightarrow E\left[X_1X_2X_3X_4\right]=\left(c_{12}c_{34}+c_{13}c_{24}+c_{14}c_{23}\right)$$

Since X_i are zero mean random variables $c_{ij}=E\left[X_iX_j\right]$

$$\rightarrow E\left[X_1X_2X_3X_4\right]=E\left[X_1X_2\right]E\left[X_3X_4\right]+E\left[X_1X_3\right]E\left[X_2X_4\right]+E\left[X_1X_4\right]E\left[X_2X_3\right]$$

Problem 6.14

$$\frac{d}{dy}F_{Y_m}\left(y\right)=\frac{d}{dy}\sum_{k=m}^{N}\binom{N}{k}\left(F_X\left(y\right)\right)^k\left(1-F_X\left(y\right)\right)^{N-k}$$
$$=\sum_{k=m}^{N}\binom{N}{k}kf_X\left(y\right)\left(F_X\left(y\right)\right)^{k-1}\left(1-F_X\left(y\right)\right)^{N-k}$$
$$-\sum_{k=m}^{N}\binom{N}{k}\left(N-k\right)f_X\left(y\right)\left(F_X\left(y\right)\right)^{k}\left(1-F_X\left(y\right)\right)^{N-k-1}$$

Since k=N evaluate the term (N-k) in the second sum to zero, we can set the range of k to be k=m, …, N-1.

$$\frac{d}{dy}F_{Y_m}(y)=\sum_{k=m}^{N}\frac{N!}{(k-1)!(N-k)!}f_X(y)(F_X(y))^{k-1}(1-F_X(y))^{N-k}$$
$$-\sum_{k=m}^{N-1}\frac{N!}{(k)!(N-k-1)!}f_X(y)(F_X(y))^{k}(1-F_X(y))^{N-k-1}$$

In the second sum, let n=k+1. Then

$$f_{Y_m}(y)=\frac{d}{dy}F_{Y_m}(y)=\sum_{k=m}^{N}\frac{N!}{(k-1)!(N-k)!}f_X(y)(F_X(y))^{k-1}(1-F_X(y))^{N-k}$$
$$\sum_{n=m+1}^{N}\frac{N!}{(n-1)!(N-n)!}f_X(y)(F_X(y))^{n-1}(1-F_X(y))^{N-n}$$

The second series cancel the first series except the terms k=m. Hence

$$f_{Y_m}(y)=\frac{N!}{(m-1)!(N-m)!}f_X(y)(F_X(y))^{m-1}(1-F_X(y))^{N-m}$$

Problem 6.17

(a)

$$E[Y_n] \;=\; E\left[\frac{X_n+X_{n-1}}{2}\right]=\frac{1}{2}E[X_n]+\frac{1}{2}E[X_{n-1}]=0.$$

$$
\begin{aligned}
E[Y_n Y_k] &= E\left[\left(\frac{X_n + X_{n-1}}{2}\right)\left(\frac{X_k + X_{k-1}}{2}\right)\right] \\
&= \frac{1}{4}\left(E[X_n X_k] + E[X_n X_{k-1}] + E[X_{n-1} X_k] + E[X_{n-1} X_{k-1}]\right) \\
&= \frac{1}{4}[c_{n,k} + c_{n,k-1} + c_{n-1,k} + c_{n-1,k-1}]
\end{aligned}
$$

Using $c_{i,j} = \delta_{i,j}$ we get

$$
E[Y_n Y_k] = \begin{cases} \frac{1}{2} & n = k, \\ \frac{1}{4} & n = k \pm 1, \\ 0 & \text{otherwise.} \end{cases}
$$

The correlation matrix is then of the form

$$
\mathbf{R} = \frac{1}{4}\begin{bmatrix} 2 & 1 & 0 & 0 & 0 & \dots \\ 1 & 2 & 1 & 0 & 0 & \dots \\ 0 & 1 & 2 & 1 & 0 & \dots \\ 0 & 0 & 1 & 2 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.
$$

(b) In this case, $c_{i,j} = \sigma_X^2 \delta_{i,j}$ so that

$$
\mathbf{R} = \frac{\sigma_X^2}{4}\begin{bmatrix} 2 & 1 & 0 & 0 & 0 & \dots \\ 1 & 2 & 1 & 0 & 0 & \dots \\ 0 & 1 & 2 & 1 & 0 & \dots \\ 0 & 0 & 1 & 2 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}.
$$