Solutions to Chapter 7 Exercises (Part 2)

Problem 7.4

Consider a discrete random variable, $X \in \{-1, 0, 1\}$, whose PMF is

$$P_X(k) = \begin{cases} \epsilon & k = \pm 1, \\ 1 - 2\epsilon & k = 0. \end{cases}$$

For this random variable, $\mu_X = 0$ and $\sigma_X^2 = 2\epsilon$. Both the sample mean and the median will be unbiased in this case. The variance of these two estimators are as follows:

Sample Mean:

$$Var(\hat{\mu}) = \frac{\sigma_X^2}{n} = \frac{2\epsilon}{n}.$$

Median: Suppose $n = 2k - 1$ samples are taken. Then Y_k is the median.

$$\begin{aligned} \Pr(Y_k = 1) &= \Pr(k \text{ or more } X\text{'s} = 1) \\ &= \sum_{m=k}^{n} \binom{n}{m} \epsilon^m (1-\epsilon)^{n-m} \\ \Pr(Y_k = -1) &= \Pr(Y_k = 1) \\ \Pr(Y_k = 0) &= 1 - 2\Pr(Y_k = 1) \\ Var(Y_k) &= 2\Pr(Y_k = 1) = 2\sum_{m=k}^{n} \binom{n}{m} \epsilon^m (1-\epsilon)^{n-m} \end{aligned}$$

Note that for small ϵ, $Var(Y_k) \sim \epsilon^{n/2}$. Hence while the variance of the sample mean decays in an inverse linear fashion with n, the variance of the median decays exponentially in n. Hence in this case, the median would give a better (lower variance) estimate of the mean.

Problem 7.5

Since X_m (m from 1 to n) are IID sequence, assume the expected value and the variance are μ and σ respectively. Moreover, since

$$\hat{\mu} = \frac{1}{n}\sum_{m=1}^{n} X_m \ ,$$

$$\begin{aligned}\hat{\sigma^2} &= \frac{1}{n}\sum_{m=1}^{n}(X_m - \hat{\mu})^2 \\ &= \frac{1}{n}\sum_{m=1}^{n} X_m^2 - \hat{\mu}^2\end{aligned}$$

From that, we have

$$\begin{aligned}E(\hat{\sigma^2}) &= \frac{1}{n}\sum_{m=1}^{n} E(X_m^2) - E(\hat{\mu}^2) \\ &= \frac{1}{n}\sum_{m=1}^{n}(\mu^2 + \sigma^2) - E(\hat{\mu}^2) \\ &= \mu^2 + \sigma^2 - \frac{1}{n^2}E[(\sum_{m=1}^{n} X_m)^2] \\ &= \mu^2 + \sigma^2 - \frac{1}{n^2}(n\sigma^2 + n^2\mu^2) \\ &= \frac{n-1}{n}\sigma^2\end{aligned}$$

Hence, this estimate is biased.

Problem 7.6

$$\begin{aligned}
\hat{\mu} &= \frac{1}{n}\sum_{m=1}^{n} X_m \\
\hat{s}^2 &= \frac{1}{n-1}\sum_{m=1}^{n}(X_m-\hat{\mu})^2 \\
Var(\hat{s}^2) &= E\left[(\hat{s}^2)^2\right]-\left(E\left[\hat{s}^2\right]\right)^2 \\
&= E\left[(\hat{s}^2)^2\right]-\sigma^4
\end{aligned}$$

We can write the equation for $\hat{s}^2$ a little differently using matrix tranformations. First make the following definitions:

$$\begin{aligned}
Y_m &= X_m-\hat{\mu} \\
Y_m &= [-\frac{1}{n}-\frac{1}{n}\cdots\frac{n-1}{n}\cdots-\frac{1}{n}][X_1\cdots X_m\cdots X_n]^T \\
\mathbf{Y} &= [Y_1,Y_2,\cdots Y_n]^T=\mathbf{AX} \\
\mathbf{A} &= \begin{bmatrix} \frac{n-1}{n} & -\frac{1}{n} & \cdots & -\frac{1}{n} \\ -\frac{1}{n} & \frac{n-1}{n} & \cdots & -\frac{1}{n} \\ \cdots & \cdots & \cdots & \cdots \\ -\frac{1}{n} & -\frac{1}{n} & \cdots & \frac{n-1}{n} \end{bmatrix}_{n\times n}
\end{aligned}$$

Note that the covariance matrix of $\mathbf{Y}$ is

$$\mathbf{C_Y}=E[\mathbf{YY}^T]=E[\mathbf{AXX}^T\mathbf{A}^T]=\mathbf{AC_XA}^T=\sigma_X^2\mathbf{AA}^T=\sigma_X^2\mathbf{A}.$$

From this we see that $Var(Y_k)=\sigma_X^2\frac{n-1}{n}$ and $Cov(Y_k,Y_m)=-\sigma_X^2/n$. Furthermore, since the Y_k are Gaussian, we have the following higher order moments (which will be needed soon):

$$\begin{aligned}
E[Y_k^4] &= 3\sigma_Y^4=3\sigma_X^4\frac{(n-1)^2}{n^2}, \\
E[Y_kY_m] &= E[Y_k^2]E[Y_m^2]+2E[Y_kY_m]^2 \\
&= \sigma_Y^4+2Cov(Y_k,Y_m)^2 \\
&= \sigma_X^4\frac{(n-1)^2}{n^2}+2\sigma_X^4\frac{1}{n^2} \\
&= \sigma_X^4\frac{n^2-2n+3}{n^2}.
\end{aligned}$$

The variance of the sample variance is then found according to

$$\begin{aligned}
E[(\hat{s}^2)^2] &= \frac{1}{(n-1)^2}\sum_{i=1}^{n}\sum_{j=1}^{n}E[Y_i^2Y_j^2] \\
&= \frac{1}{(n-1)^2}\left\{nE[Y_i^4]+(n^2-n)E[Y_i^2Y_j^2]\right\} \\
&= \frac{1}{(n-1)^2}\left\{3n\sigma_X^4\frac{(n-1)^2}{n^2}+(n^2-n)\sigma_X^4\frac{n^2-2n+3}{n^2}\right\} \\
&= \sigma_X^4\frac{n+1}{n-1}.
\end{aligned}$$

$$\begin{aligned}
Var(\hat{s}^2) &= E[(\hat{s}^2)^2]-E[\hat{s}^2]^2 \\
&= \sigma_X^4\frac{n+1}{n-1}-\sigma_X^4 \\
&= \frac{2}{n-1}\sigma_X^4.
\end{aligned}$$

Problem 7.9

(a) A sequence converges in the mean square sense(MSS) if

$$\lim_{n\to\infty}E\left[|S_n-S|^2\right] = 0, \qquad (1)$$

and it converges in probability if

$$\lim_{n\to\infty}Pr\left(|S_n-S|>\epsilon\right) = 0. \qquad (2)$$

Applying Markov's ineqaulity to the LHS of (2) we get

$$\begin{aligned}
\lim_{n\to\infty}Pr\left(|S_n-S|>\epsilon\right) &\le \frac{lim_{n\to\infty}E\left[|S_n-S|^2\right]}{\epsilon^2} \\
&\le 0 \text{ (since the sequence converges in MSS)} \\
\Rightarrow \lim_{n\to\infty}Pr\left(|S_n-S|>\epsilon\right) &= 0 \text{ (since probabilities cannot be negative)}
\end{aligned}$$

Problem 7.16

If M is an exponential random variable, then $E[M] = \mu_M$ and $Var(M) = \sigma_M^2 = \mu_M^2$. It is desired that the confidence interval have a width of $\epsilon = 0.2\mu_M$. Hence, the number of samples is determined from $\epsilon = c_{0.9}\sigma_{\hat{\mu}}$. This results in

$$\begin{aligned} 0.2\mu_M &= 1.64\sigma_M/\sqrt{n} = 1.64\mu_M/\sqrt{n} \\ \Rightarrow n &= 67.24. \end{aligned}$$

At least 68 failures need to be observed.

Refer to page 258 (Confidence Intervals), equations 7.38 and 7.39.

Problem 7.17

$$E[S_N] = E\left[\sum_{i=1}^{\infty} Y_i Z_i\right] = \sum_{i=1}^{\infty} E[Y_i Z_i]$$

Note the value of Y_i is determine by whether or not the test terminates before time$i$. In particular, the values of $\{Z_1, Z_2, \ldots, Z_{i-1}\}$ determine Y_i. In other words, Y_i is dependent on $\{Z_1, Z_2, \ldots, Z_{i-1}\}$ but not on $\{Z_i, Z_{i+1}, \ldots\}$. Therefore, Y_i and Z_i are independent.

$$\begin{aligned} \Rightarrow E[Y_i Z_i] &= E[Y_i]E[Z_i]. \\ E[S_N] &= \sum_{i=1}^{\infty} E[Y_i]E[Z_i] \end{aligned}$$

Note that Z_i is independent of i since the Z_i are IID. Hence,

$$\begin{aligned} E[S_N] &= E[Z_i]\sum_{i=1}^{\infty} E[Y_i] \\ &= E[Z_i]E\left[\sum_{i=1}^{\infty} Y_i\right] \\ &= E[Z_i]E\left[\sum_{i=1}^{N} 1\right] \\ &= E[Z_i]E[N]. \end{aligned}$$