MVE135 Random Processes with Applications Fall 2010

Written exam Monday 15 August 2011 8.30 am - 12.30 am

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AIDS: Beta.

GRADES: 12 (40%), 18 (60%) and 24 (80%) points for grade 3, 4 and 5, respectively.

GOOD LUCK!

Task 1. Find analytic explicit expressions for the value of the integral $\int_{-\infty}^{\infty} \exp\big(ax^2 + bx\big)\, dx$ for all different values of the constants $a, b \in \mathbb{R}$. **(5 points)**

Task 2. Let X and Y be independent zero-mean and unit-variance Gaussian random variables. Find the PDF of the random variable (U, V) given by $U = X\cos(\theta) - Y\sin(\theta)$ and $V = X\sin(\theta) + Y\cos(\theta)$ for a constant $\theta \in \mathbb{R}$. **(5 points)**

Task 3. Find the conditional probability $P[X(1) = 1 \,|\, X(2) = 2]$ for a Poisson process $X(t)$ with intensity $\lambda > 0$. **(5 points)**

Task 4. Find a stationary process that is not WSS. Find a WSS process that is not stationary. **(5 points)**

Task 5. A continuous time WSS process $X(t)$ with auto-correlation function $R_{XX}(\tau)$ is filtered through a filter with impulse response $h_1(t)$. The output of this filter $Y(t)$ is filtered through a second filter with impulse response $h_2(t)$ that has output $Z(t)$. Find the cross-correlation function between $X(t)$ and $Z(t)$. **(5 points)**

Task 6. In a digital communication system either a known deterministic signal $s(t)$ is sent (representing 1) or the zero signal is sent (representing 0). The sent signal travels on a noisy channel where a Gaussian white noise disturbance process $N(t)$ with PSD $N_0/2$ is added. The recived signal is thus either $X(t) = s(t) + N(t)$ (if $s(t)$ is sent) or $X(t) = N(t)$ (if 0 is sent). The task of the reciver is to at a certain time t_0 try to decide whether it is $s(t)$ ot 0 that has been sent. That is done by first filtering the recived signal through a filter with impulse response $h(t)$, the output of which is $Y(t) = X(t) \star h(t)$, and then decide that $s(t)$ is sent if $Y(t) \geq \frac{1}{2}\, s(t) \star h(t)$ and that 0 is sent if $Y(t) < \frac{1}{2}\, s(t) \star h(t)$. The two decision error probabilities are defined as

$$P[\text{decides } s(t) \text{ sent when 0 sent}] = P\big[N(t) \star h(t) \geq \tfrac{1}{2}\, s(t) \star h(t)\big]$$

and

$$P[\text{decides 0 sent when } s(t) \text{ sent}] = P\big[s(t) \star h(t) + N(t) \star h(t) < \tfrac{1}{2}\, s(t) \star h(t)\big]$$
$$= P\big[N(t) \star h(t) < -\tfrac{1}{2}\, s(t) \star h(t)\big].$$

Show that these error probabilities have a common smallest possible value

$$1 - \Phi\left(\sqrt{\frac{\int_{-\infty}^{\infty} s(t)^2\,dt}{2\,N_0}}\right)$$

when $h(t)$ is choosen optimally (to minimize these probabilities). **(5 points)**

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Solutions to written exam Monday 15 August 2011

Task 1. For $a>0$ we have $\mathrm{e}^{ax^2+bx}\to\infty$ as $x\to\pm\infty$ so that the value of the integral is ∞. For $a=0$ we have $\mathrm{e}^{bx}\to\infty$ as $x\to\infty$ if $b>0$, $\mathrm{e}^{bx}=1$ for all x if $b=0$, while $\mathrm{e}^{bx}\to\infty$ as $x\to-\infty$ if $b<0$ so that again the integral equals ∞. For $a<0$ the value of the integral is $\sqrt{-\pi/a}\,\mathrm{e}^{-b^2/(4a)}$ by Exercise 3.9 in the book of Miller and Childers.

Task 2. The random variables U and V are independent zero-mean and unit-variance Gaussian by Exercise 5.28 in the book of Miller and Childers.

Task 3. We have $P[X(1)=1\,|\,X(2)=2] = P[X(1)=1,\,X(2)=2]/P[X(2)=2] = P[X(1)=1,\,X(2)-X(1)=1]/P[X(2)=2] = P[X(1)=1]\,P[X(2)-X(1)=1]/P[X(2)=2] = (P[X(1)=1])^2/P[X(2)=2] = ((\lambda^1/(1!))\,\mathrm{e}^{-\lambda})^2/(((2\lambda)^2/(2!))\,\mathrm{e}^{-2\lambda}) = 1/2$.

Task 4. A stationary process with infinite variance is not WSS. A sequence of zero-mean unit-variance uncorrelated random variables is a discrete time WSS process, but is not stationary unless the CDF of all members of the sequence coincide.

Task 5. We have

$$\begin{aligned}
R_{XZ}(\tau) &= E[X(t)Z(t+\tau)]\\
&= E\bigg[X(t)\int_{-\infty}^{\infty} Y(t+\tau-s)\,h_2(s)\,ds\bigg]\\
&= E\bigg[X(t)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} X(t+\tau-s-r)\,h_2(s)h_1(r)\,dsdr\bigg]\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E[X(t)X(t+\tau-s-r)]\,h_2(s)h_1(r)\,dsdr\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{XX}(\tau-s-r)\,h_2(s)h_1(r)\,dsdr\\
&= R_{XX}(\tau)\star h_1(\tau)\star h_2(\tau).
\end{aligned}$$

Task 6. As $N(t)\star h(t)$ is zero-mean Gaussian it is clear from symmetry that the two error probabilities agree with a common value

$$1-\Phi\Bigg(\frac{\frac{1}{2}\,s(t)\star h(t)}{\sqrt{\mathrm{Variance}[N(t)\star h(t)]}}\Bigg) = 1-\Phi\Bigg(\frac{\frac{1}{2}\,s(t)\star h(t)}{\sqrt{N_0\int_{-\infty}^{\infty}s(t)^2\,dt/2}}\Bigg).$$

This error probability is minimized when the argument of the function $1-\Phi(x)$ is maximized. By the theory for the matched filter the maximal valued of that argument is the argument claimed in the task, which in turn is achived for $h(t)=s(t_0-t)$.