MVE135 Random processes with applications

Written exam Monday 20 August 2012 8.30 am – 12.30 pm

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AIDS: Beta.

GRADES: 12, 18 and 24 points for grades 3, 4 and 5, respectively. GOOD LUCK!

Task 1. Let X be a Gaussian random variable such that $X \sim \mathrm{N}(0,1)$. Find the conditional PDF $f_{X^2 \,\big|\, |X|\leq 3}(x)$. **(5 points)**

Task 2. A pair of random variables has a joint PDF specified by $f_{X,Y}(x,y) = \frac{1}{2\pi\sqrt{3}} \exp\left(-\frac{x^2+2xy+4y^2}{6}\right)$. Find the conditional PDF $f_{X|Y}(x|y)$. **(5 points)**

Task 3. Show by example that the random process $Z(t) = X(t) + Y(t)$ may be a wide sense stationary process even though the random processes $X(t)$ and $Y(t)$ are not.

(5 points)

Task 4. Let $X(t)$ be a continuous-time random process with power spectral density $S_{XX}(f)$. The derivative process of $X(t)$ is defined as $X'(t) = \lim_{h\to 0}(X(t+h)-X(t))/h$ whenever this limit is well-defined in a suitable sense. Show that the cross spectral density between $X(t)$ and $X'(t)$ is given by $S_{XX'}(f) = j2\pi f\, S_{XX}(f)$. **(5 points)**

Task 5. Let a pair of zero-mean jointly Gaussian continuous time processes $X_1(t)$ and $X_2(t)$ be inputs to linear filters with impulse responses $h_1(t)$ and $h_2(t)$, respectively, and corresponding outputs $Y_1(t)$ and $Y_2(t)$. Under what exact (i.e., necessary and sufficient) conditions on the crosscorrelation function $R_{X_1X_2}(t_1,t_2) = \mathbf{E}\{X_1(t_1)X_2(t_2)\}$ are two output values $Y_1(s)$ and $Y_2(t)$ independent? **(5 points)**

Task 6. Suppose $x[n]$ is an AR(1) process with $a_1 = 0.5$. Derive and illustrate (plot) the power spectral density of $x[n]$, assuming that the white noise input has variance 2.

(5 points)

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Solutions to written exam Monday 20 August 2012

Task 1. As $F_{X^2\,\big|\,|X|\leq 3}(x) = Pr\big(X^2\leq x,\,|X|\leq 3\big)/Pr(|X|\leq 3) = Pr\big(|X|\leq\sqrt{x},\,|X|\leq 3\big)/Pr(|X|\leq 3) = Pr(|X|\leq\sqrt{x}\,)/Pr(|X|\leq 3) = \big(\int_{-\sqrt{x}}^{\sqrt{x}}\frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-y^2/2}\,dy\big)/(\Phi(3)-\Phi(-3))$ for $0\leq x\leq 9$ and $F_{X^2\,\big|\,|X|\leq 3}(x) = Pr\big(X^2\leq x,\,|X|\leq 3\big)/Pr(|X|\leq 3) = Pr(|X|\leq 3)/Pr(|X|\leq 3) = 1$ for $x>9$, we have $f_{X^2\,\big|\,|X|\leq 3}(x) = \frac{d}{dx}F_{X^2\,\big|\,|X|\leq 3}(x) = \frac{1}{\sqrt{2\pi x}}\,\mathrm{e}^{-x/2}/(\Phi(3)-\Phi(-3))$ for $0\leq x\leq 9$ and $f_{X^2\,\big|\,|X|\leq 3}(x) = 0$ for $x>9$.

Task 2. As $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-y^2/2}$ for $y\in\mathbb{R}$, we have $f_{X|Y}(x|y) = f_{X,Y}(x,y)/f_Y(y) = \frac{1}{\sqrt{6\pi}}\,\mathrm{e}^{-(x+y)^2/6}$ for $x\in\mathbb{R}$.

Task 3. This is one of the home exercises listed for the course. For example, given any wide sense stationary process $Z(t)$ we may take $X(t) = Z(t)/2-t$ and $Y(t) = Z(t)/2+t$.

Task 4. As $R_{XX'}(\tau) = \mathbf{E}\big\{X(t)\lim_{h\to 0}(X(t+\tau+h)-X(t+\tau))/h\big\} = \lim_{h\to 0}\mathbf{E}\big\{X(t)(X(t+\tau+h)-X(t+\tau))/h\big\} = \lim_{h\to 0}(R_{XX}(\tau+h)-R_{XX}(\tau))/h = R'_{XX}(\tau)$, we have $S_{XX'}(f) = \int_{-\infty}^{\infty} R_{XX'}(\tau)\,\mathrm{e}^{-j2\pi f\tau}\,d\tau = \int_{-\infty}^{\infty} R'_{XX}(\tau)\,\mathrm{e}^{-j2\pi f\tau}\,d\tau = j2\pi f\int_{-\infty}^{\infty} R_{XX}(\tau)\,\mathrm{e}^{-j2\pi f\tau}\,d\tau = j2\pi f\,S_{XX}(f)$.

Task 5. As the outputs $Y_1(s)$ and $Y_2(t)$ are jointly Gaussian, the exact condition is that their crosscorrelation is zero, which is to say that $R_{Y_1Y_2}(s,t) = \mathbf{E}\big\{\big(\int_{-\infty}^{\infty} h_1(u)X_1(s-u)\,du\big)\big(\int_{-\infty}^{\infty} h_2(v)X_2(t-v)\,dv\big)\big\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(u)h_2(v)\,\mathbf{E}\{X_1(s-u)X_2(t-v)\}\,dudv = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(u)h_2(v)\,R_{X_1X_2}(s-u,t-v)\,dudv = 0$.

Task 6. It is given that

$$x[n] + 0.5x[n-1] = e[n],$$

where $e[n]$ is a white noise process such that $E\{e[n]^2\} = 2$. An equivalent description is that $x[n]$ is the output from a linear system, with the transfer function

$$H(z) = \frac{1}{1+0.5z^{-1}},$$

where $e[n]$ is the input signal. The PSD of $x[n]$ is therefore

$$\begin{aligned} P_x\left(e^{j\omega}\right) &= \left|H\left(e^{j\omega}\right)\right|^2 P_e\left(e^{j\omega}\right) \\ &= \frac{\sigma_e^2}{\left|1+0.5e^{-j\omega}\right|^2} = \frac{2}{\left|1+0.5e^{-j\omega}\right|^2}. \end{aligned}$$

In order to plot (sketch) $P_x\left(e^{j\omega}\right)$ it helps to notice that $H(z)$ has a pole in $z = -0.5$, since this tells us that it is a high-pass filter. Further, it is easy to see that $P_x\left(e^{j\pi}\right) = 8$. A detailed plot of PSD is given in Figure 1 below.

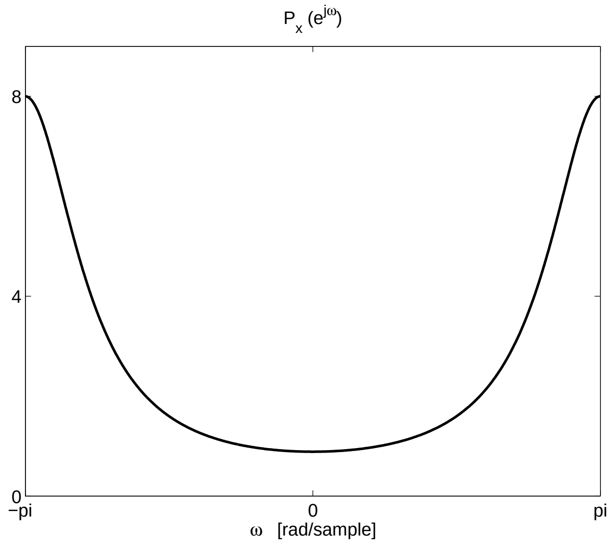

Figure 1: The power spectral density of $x[n]$.