MVE135 Random Processes with Applications

Written Exam Thursday 17 January 2013 2 pm – 6 pm

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Aids: Beta.

Grades: 12 (40%), 18 (60%) and 24 (80%) points for grade 3, 4 and 5, respectively.

Task 1. Find $E[X\,|\,X>1]$ for an exponentially distributed random variable X with $E[X]=1$. **(5 points)**

Task 2. Find the PDF of $Z=\sqrt{X^2+Y^2}$ when X and Y are independent standard normal random variables. **(5 points)**

Task 3. Let $X(t)$ and $Y(t)$ be independent Poisson processes, both with rates 1. Define $Z(t)=X(t)+Y(t)$. Find $E[X(1)\,|\,Z(2)=2]$. **(5 points)**

Task 4. Let $X(t)$ be a Poisson process with mean function $\mu_X(t)=\lambda\,t$ and autocovariance function $C_{XX}(s,t)=\lambda\,\min(s,t)$. Show that the process $Y(t)=(X(t)-\lambda\,t)/\sqrt{t}$ is so called group-WSS with respect to multiplication on $(0,\infty)$, which is to say that $E[Y(ht)]=E[Y(t)]$ and $E[Y(hs)Y(ht)]=E[Y(s)Y(t)]$ for $h,s,t>0$. **(5 points)**.

Task 5. Let $X(t)$ be a continuous-time strict sense stationary Gaussian process with zero mean and autocorrelation function $R_{XX}(\tau)=\mathrm{e}^{-|\tau|}$. Show that the process $Y(t)=\mathrm{e}^{X(t)^2}$ is strict sense stationary but not WSS. **(5 points)**

Task 6. A WSS discrete-time random process $X(n)$ with PSD $S_{XX}(f)$ is input to two different LTI systems with transfer functions $H_1(f)$ and $H_2(f)$, respectively. Find the cross spectral density $S_{Y_1Y_2}(f)$ between the outputs $Y_1(n)$ and $Y_2(n)$ from the two LTI systems. **(5 points)**

Good Luck!

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Solutions to Written Exam Thursday 17 January 2013

Task 1. As $f_X(x) = \mathrm{e}^{-x}$ for $x \geq 0$ we have $f_{X|X>1}(x) = f_X(x)/Pr(X > 1)$ for $x > 1$, where $Pr(X > 1) = \int_1^\infty \mathrm{e}^{-x}\,dx = \mathrm{e}^{-1}$, so that $E[X\,|\,X > 1] = \int_1^\infty x\,\mathrm{e}^{1-x}\,dx = \int_1^\infty (x-1)\,\mathrm{e}^{1-x}\,dx + \int_1^\infty \mathrm{e}^{1-x}\,dx = E[X] + 1 = 2$.

Task 2. We have $f_Z(z) = \frac{d}{dz}P(Z \leq z) = \frac{d}{dz}P(\sqrt{X^2+Y^2} \leq z) = \frac{d}{dz}\iint_{\{(x,y)\in\mathbb{R}^2:\sqrt{x^2+y^2}\leq z\}} f_{X,Y}(x,y)\,dxdy = \frac{d}{dz}\iint_{\{(x,y)\in\mathbb{R}^2:\sqrt{x^2+y^2}\leq z\}} \frac{1}{2\pi}\,\mathrm{e}^{-(x^2+y^2)/2}\,dxdy = \frac{d}{dz}\int_{r=0}^{r=z}\int_{\theta=0}^{\theta=2\pi} \frac{1}{2\pi}\,\mathrm{e}^{-r^2/2}\, r\,drd\theta = \frac{d}{dz}(1-\mathrm{e}^{-z^2/2}) = z\,\mathrm{e}^{-z^2/2}$ for $z \geq 0$.

Task 3. We have $P(X(1) = k\,|\,Z(2) = 2) = P(X(1) = k,\, X(2)+Y(2) = 2)/P(X(2) + Y(2) = 2) = [P(X(1) = k,\, X(2)+Y(2) = 2,\, Y(2) = 0) + P(X(1) = k,\, X(2)+Y(2) = 2,\, Y(2) = 1) + P(X(1) = k,\, X(2)+Y(2) = 2,\, Y(2) = 2)]/[P(X(2) = 2,\, Y(2) = 0) + P(X(2) = 1,\, Y(2) = 1) + P(X(2) = 0,\, Y(2) = 2)] = [P(X(1) = k)\,P(X(2)-X(1) = 2-k)\,P(Y(2) = 0) + P(X(1) = k)\,P(X(2)-X(1) = 1-k)\,P(Y(2) = 1) + P(X(1) = k)\,P(X(2)-X(1) = -k)\,P(Y(2) = 2)]/[\frac{1}{2}\,\mathrm{e}^{-2}+\mathrm{e}^{-2}+\frac{1}{2}\,\mathrm{e}^{-2}] = \frac{1}{2}\,\mathrm{e}\,[P(X(1) = k)\,P(X(1) = 2-k) + P(X(1) = k)\,P(X(1) = 1-k) + P(X(1) = k)\,P(X(1) = -k)\,\frac{1}{2}] = \mathrm{e}^{-1}\,[\delta(k) + \delta(k-1) + \frac{1}{4}\,\delta(k-2)]$, so that $E[X(1)\,|\,Z(2) = 2] = \mathrm{e}^{-1}\,[1\cdot 0 + 1\cdot 1 + \frac{1}{4}\cdot 2] = \frac{3}{2}\,\mathrm{e}^{-1}$.

Task 4. We have $E[Y(ht)] = E[(X(ht) - \lambda\,ht)/\sqrt{ht}] = (\mu_X(ht) - \lambda\,ht)/\sqrt{ht} = 0 = E[Y(t)]$ (where the last equality follows from taking $h = 1$) and $E[Y(hs)Y(ht)] = C_{XX}(hs,ht)/\sqrt{h^2st} = \lambda\,\min(hs,ht)/\sqrt{h^2st} = \lambda\,\min(\sqrt{s/t}\,,\sqrt{t/s}\,) = E[Y(s)Y(t)]$.

Task 5. As $X(t)$ is strict sense stationary we have $P(Y(t_1+h) \leq x_1,\ldots,Y(t_n+h) \leq x_n) = 0 = P(Y(t_1) \leq x_1,\ldots,Y(t_n) \leq x_n)$ if $\min(x_1,\ldots,x_n) \leq 0$ while $P(Y(t_1+h) \leq x_1,\ldots,Y(t_n+h) \leq x_n) = P(-\sqrt{\ln(x_1)} \leq X(t_1+h) \leq \sqrt{\ln(x_1)}\,,\ldots,-\sqrt{\ln(x_n)} \leq X(t_n+h) \leq \sqrt{\ln(x_n)}\,) = P(-\sqrt{\ln(x_1)} \leq X(t_1) \leq \sqrt{\ln(x_1)}\,,\ldots,-\sqrt{\ln(x_n)} \leq X(t_n) \leq \sqrt{\ln(x_n)}\,) = P(Y(t_1) \leq x_1,\ldots,Y(t_n) \leq x_n)$ otherwise, so that also $Y(t)$ is strict sense stationary. However, as $\mu_Y(t) = E[Y(t)] = E[\mathrm{e}^{X(t)^2}] = \int_{-\infty}^\infty \mathrm{e}^{x^2}\,\frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-x^2/2}\,dx = \infty$ does not exist $Y(t)$ is not WSS.

Task 6. We have $S_{Y_1Y_2}(f) = \sum_{\tau=-\infty}^\infty \mathrm{e}^{-j2\pi f\tau}\,E[\sum_{k=-\infty}^\infty h_1(k)X(n-k)\sum_{\ell=-\infty}^\infty h_2(\ell)\,X(n+\tau-\ell)] = \sum_{k=-\infty}^\infty\sum_{\ell=-\infty}^\infty \mathrm{e}^{-j2\pi f(\ell-k)}h_1(k)h_2(\ell)\sum_{\tau=-\infty}^\infty \mathrm{e}^{-j2\pi f(\tau-\ell+k)}R_{XX}(\tau-\ell+k) = \overline{H_1(f)}H_2(f)S_{XX}(f)$.