MVE135 Random Processes with Applications

Written Exam Monday 19 August 2013 2 pm – 6 pm

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AIDS: Beta.

GRADES: 12 (40%), 18 (60%) and 24 (80%) points for grade 3, 4 and 5, respectively.

GOOD LUCK!

Task 1. Find the conditional probability $\Pr(X < 3|X > 2)$ for a continuous random variable X with PDF $f_X(x) = c/(x^2+4)$. (Here c is a positive constant that is determined by the fact that $f_X(x)$ is a PDF.) **(5 points)**

Task 2. Let X and Y be zero-mean, unit variance Gaussian random variables with correlation coefficient ρ. Suppose we form two new random variables U and V using a linear trasnformation as

$$U = a\,X + b\,Y \quad \text{and} \quad V = c\,X + d\,Y,$$

where a, b, c and d are real (non-random) coefficients/constants. Find constraints on a, b, c and d such that U and V are independent. **(5 points)**

Task 3. Find the probability $Pr(X(1)+X(2)+X(3) > 6)$ for a continuous time WSS Gaussian process $X(t)$ with mean $\mu_X = 1$ and autocorrelation function $R_{XX}(\tau) = \mathrm{e}^{-|\tau|}+1$ for $\tau \in \mathbb{R}$. **(5 points)**

Task 4. Find the probability $Pr(X(1)+X(2) > 3)$ for a Poisson process with rate 1. **(5 points)**

Task 5. The PSD $S_{XX}(f)$ of a continuous time WSS process $X(t)$ has the properties to be real and symmetric (=even). Prove one of these properties. **(5 points)**

Task 6. Give an example of a continuous time impulse response function h that has the property that when a continuous time white noise process $N(t)$ with PSD $S_{NN}(f) = N_0/2$ is input to an LTI system with this impulse response, then the output process $Y(t)$ from the system has PSD $S_{YY}(f)$ that is decreasing for $f \geq 0$ and satisfies $S_{YY}(f_0) = S_{YY}(0)/2 = N_0/4$ for a certain frequency $f_0 > 0$. (In other words the LTI system is a lowpass filter with 3dB bandwith f_0.) **(5 points)**

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Solutions to Written Exam Monday 19 August 2013

Task 1. This is a part of Exercise 3.5 in the book by Miller and Childers. The solution is available at the URL

`www.math.chalmers.se/Stat/Grundutb/CTH/mve135/1011/Exercises/HomeExercise2.pdf`

Task 2. This is Exercise 5.29 in the book by Miller and Childers. The solution is available at the URL

`www.math.chalmers.se/Stat/Grundutb/CTH/mve135/1011/Exercises/HomeExercise4.pdf`

Task 3. As $X(1)+X(2)+X(3)$ is $N(m,\sigma^2)$-distributed we have $Pr(X(1)+X(2)+X(3) > 6) = Pr(N(m,\sigma^2) > 6) = 1-\Phi((6-m)/\sigma)$, where $m = E[X(1)+X(2)+X(3)] = 3$ and $\sigma^2 = \mathrm{Var}(X(1)+X(2)+X(3)) = 3\,C_{XX}(0)+4\,C_{XX}(1)+2\,C_{XX}(2) = 3+4\,\mathrm{e}^{-1}+2\,\mathrm{e}^{-2}$ [using that $C_{XX}(\tau) = R_{XX}(\tau)-\mu_X^2 = \mathrm{e}^{-|\tau|}$].

Task 4. As $X(1)+X(2) = (X(2)-X(1))+2\,X(1)$ where $X(2)-X(1)$ and $X(1)$ are independent Po(1)-distributed we have $Pr(X(1)+X(2) > 3) = Pr((X(2)-X(1))+2\,X(1) > 3) = Pr(X(1)\geq 2)+Pr(X(1)=1,\, X(2)-X(1)>1)+Pr(X(1)=0,\, X(2)-X(1)>3) = (1-\mathrm{e}^{-1}-\mathrm{e}^{-1})+\mathrm{e}^{-1}(1-\mathrm{e}^{-1}-\mathrm{e}^{-1})+\mathrm{e}^{-1}(1-\mathrm{e}^{-1}-\mathrm{e}^{-1}-\frac{1}{2}\mathrm{e}^{-1}-\frac{1}{6}\mathrm{e}^{-1})$.

Task 5. As the autocorrelation function $R_{XX}(\tau)$ is symmetric we have $S_{XX}(-f) = \int_{-\infty}^{\infty} R_{XX}(\tau)\,\mathrm{e}^{-j2\pi(-f)\tau}\,d\tau = \int_{-\infty}^{\infty} R_{XX}(\tau)\,\mathrm{e}^{-j2\pi f(-\tau)}\,d\tau = \int_{-\infty}^{\infty} R_{XX}(-\hat{\tau})\,\mathrm{e}^{-j2\pi f\hat{\tau}}\,d\hat{\tau} = \int_{-\infty}^{\infty} R_{XX}(\hat{\tau})\,\mathrm{e}^{-j2\pi f\hat{\tau}}\,d\hat{\tau} = S_{XX}(f)$ and $\overline{S_{XX}(f)} = \overline{\int_{-\infty}^{\infty} R_{XX}(\tau)\,\mathrm{e}^{-j2\pi f\tau}\,d\tau} = \int_{-\infty}^{\infty} R_{XX}(\tau)\,\mathrm{e}^{j2\pi f\tau}\,d\tau =$ [see above] $= S_{XX}(f)$.

Task 6. If we take $h(t) = \exp(-t/t_0)\,u(t)/t_0$ then we have $H(f) = F[h(t)] = 1/(1+j2\pi f t_0)$, so that $S_{YY}(f) = |H(f)|^2 S_{NN}(f) = (N_0/2)/(1+4\pi^2 f^2 t_0^2)$ satisfies the imposed requirements when $t_0 = 1/(2\pi f_0)$.