Solutions to Chapter 8 Exercises

Problem 8.5

(a)

$R_{Y,Y}[n_1, n_2] = E[(X[n_1]+c)(X[n_2]+c)] = R_{X,X}[n_1, n_2]+c\mu_X[n_1]+c\mu_X[n_2]+c^2$

Since $X[n]$ is WSS, $\mu_X[n] = \mu_X$ and $R_{X,X}[n_1, n_2] = R_{X,X}[n_2 - n_1]$.

$$\Rightarrow R_{Y,Y}[n_1, n_2] = R_{X,X}[n_2 - n_1] + 2c\mu_X + c^2.$$

(b)

$$\begin{aligned} E[X[n_1]Y[n_2]] &= E[X[n_1](X[n_2] + c)] = R_{X,X}[n_2 - n_1] + c\mu_X. \\ E[X[n_1]]E[Y[n_2]] &= \mu_X(\mu_X + c) = \mu_X^2 + c\mu_X. \end{aligned}$$

The processes are not orthogonal (since $R_{X,Y}[n_1, n_2] \neq 0$).
The processes are not uncorrelated (since $R_{X,Y}[n_1, n_2] \neq \mu_x\mu_Y$).
The processes are not independent (since not uncorrelated and since $Y[n] = X[n] + c$).

Problem 8.7

(a)

$$\mu_X(t) = \mu_A \cos(\omega t) + \mu_B \sin(\omega t) = 0.$$

(b)

$$\begin{aligned} R_{X,X}(t_1, t_2) &= E[A^2]\cos(\omega t_1)\cos(\omega t_2) + E[B^2]\sin(\omega t_1)\sin(\omega t_2) \\ &+ E[AB]\cos(\omega t_1)\sin(\omega t_2) + E[AB]\sin(\omega t_1)\cos(\omega t_2) \\ &= \frac{E[A^2] + E[B^2]}{2}\cos(\omega(t_2 - t_1)) + \frac{E[A^2] - E[B^2]}{2}\cos(\omega(t_1 + t_2)). \end{aligned}$$

(c) $X(t)$ will be WSS if $E[A^2] = E[B^2] \Rightarrow \sigma_A^2 = \sigma_B^2$.

Problem 8.11

(a) Since T is uniformly distributed over one period of $s(t)$, for any time instant t, $X(t) = s(t-T)$ will be equally likely to take on any of the values in one period of $s(t)$. Since $s(t)$ is 1 half of the time and -1 half of the time, we get

$$\Pr(X(t)=1) = \Pr(X(t)=-1) = \frac{1}{2}.$$

(b)

$$E[X(t)] = (1)\cdot \Pr(X(t)=1) + (-1)\cdot \Pr(X(t)=-1) = 0.$$

This can also be seen in an alternative manner:

$$E[X(t)] = E[s(t-T)] = \int s(t-u) f_T(u) du = \int_0^1 s(t-u) du.$$

Since the integral is over one period of $s(t)$, $E[X(t)]$ is just the d.c. value (time average) of $s(t)$ which is zero.

(c)

$$\begin{aligned} R_{X,X}(t_1,t_2) &= E[s(t_1-T)s(t_2-T)] \\ &= \int_0^1 s(t_1-u)s(t_2-u)du \\ &= \int_0^1 s(v)s(v+t_2-t_1)dv \\ &= s(t) * s(-t)\Big|_{t=t_2-t_1}. \end{aligned}$$

This is the time correlation of a square wave with itself which will result in the periodic triangle wave shown in Figure 1.

(d) The process is WSS.

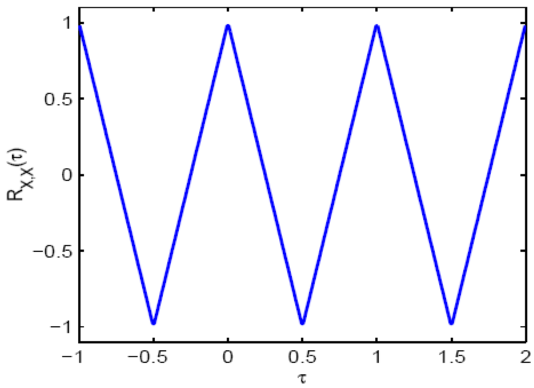

Figure 1: Autocorrelation function for process of Exercise 8.11

Problem 8.14

(a)

$$f_X(x;t) = \frac{f_A(a)}{\left|\frac{dX}{dA}\right|}\Bigg|_{A=-\frac{1}{t}\ln(x)} = \frac{f_A(-\frac{1}{t}\ln(x))}{tx}$$

(b)

$$E[X(t)] = E[e^{-At}] = \int_0^\infty e^{-at}e^{-a}da = \frac{1}{1+t}.$$

$$R_{X,X}(t_1,t_2) = E[X(t_1)X(t_2)] = E[e^{-A(t_1+t_2)}] = \frac{1}{1+t_1+t_2}.$$

The process is not WSS.

Problem 8.22

(a) Consider a time instant, t, such that $0 < t < t_o$.

$$\begin{aligned}
\Pr(X(t)=1|X(t_o)=1) &= \frac{\Pr(X(t_o)=1|X(t)=1)\Pr(X(t)=1)}{\Pr(X(t_o)=1)} \\
&= \frac{\lambda t e^{-\lambda t}}{\lambda t_o e^{-\lambda t_o}} \Pr(\text{no arrivals in } (0,t)) \\
&= \frac{t}{t_o} \exp(-\lambda(t-t_o)) \exp(-\lambda(t_o-t)) \\
&= \frac{t}{t_o}.
\end{aligned}$$

Let S_1 be the arrival time of the first arrival. Then $\{X(t)=1\} \Leftrightarrow \{S_1 \leq t\}$. Hence, given that there is one arrival in $(0, t_o)$, that is $X(t_o) = 1$,

$$\Pr(X(t)=1|X(t_o)=1) = \Pr(S_1 \leq t|X(t_o)=1) = F_{S_1}(t|X(t_o)=1) = \frac{t}{t_o}$$

$$\Rightarrow f_{S_1}(t|X(t_o)=1) = \frac{1}{t_o}, \qquad 0 \leq t < t_o.$$

(b) Let $0 \leq t_1 \leq t_2 \leq t_o$. Also define

$$\begin{aligned}
S_1 &= \text{arrival time of first arrival,} \\
S_2 &= \text{arrival time of second arrival.}
\end{aligned}$$

The joint distribution of the two arrival times is found according to:

$$\begin{aligned}
f_{S_1,S_2}(t_1,t_2|X(t_o)=2) &= f_{S_1|S_2}(t_1|S_2=t_2, X(t_o)=2) f_{S_2}(t_2|X(t_o)=2) \\
&= f_{S_1|S_2}(t_1|S_2=t_2) f_{S_2}(t_2|X(t_o)=2)
\end{aligned}$$

To find $f_{S_2}(t_2)$, proceed as in part (a).

$$\begin{aligned}
F_{S_2}(t_2|X(t_o)=2) &= \Pr(X(t_2)=2|X(t_o)=2) \\
&= \Pr(X(t_o)=2|X(t_2)=2) \frac{\Pr(X(t_2)=2)}{\Pr(X(t_o)=2)} \\
&= \exp(-\lambda(t_o-t_2)) \frac{\frac{(\lambda t_2)^2}{2} e^{-\lambda t_2}}{\frac{(\lambda t_0)^2}{2} e^{-\lambda t_o}} \\
&= \left(\frac{t_2}{t_o}\right)^2. \\
\Rightarrow f_{S_2}(t_2|X(t_o)=2) &= \frac{2t_2}{t_o^2}, \qquad 0 \leq t_2 \leq t_o.
\end{aligned}$$

Given $S_2 = t_2$ there is one arrival between 0 and t_2. From the results of part (a), we know S_1 is uniform over $(0, t_2)$ given $S_2 = t_2$. Therefore

$$f_{S_1|S_2}(t_1|t_2) = \frac{1}{t_2}, \qquad 0 \le t_1 \le t_2.$$

Putting the two previous results together we get

$$\begin{aligned} f_{S_1,S_2}(t_1, t_2|X(t_o) = 2) &= f_{S_1|S_2}(t_1|S_2 = t_2) f_{S_2}(t_2|X(t_o) = 2) \\ &= \frac{2t_2}{t_o^2} \cdot \frac{1}{t_2} \\ &= \frac{2}{t_o^2}, \qquad 0 \le t_1 \le t_2 \le t_o. \end{aligned}$$

The two arrival times S_1 and S_2 are uniformly distributed over $0 \le t_1 \le t_2 \le t_o$.

In General we can write:

$$f_{s_1, s_2, \ldots, s_n}\left(t_1, t_2, \ldots, t_n \middle| X\left(t_0\right) = n\right) = \frac{n!}{t_0^n}$$

Problem 8.23

$$\begin{aligned}
\Pr(N(t)=k|N(t+\tau)=m) &= \Pr(N(t+\tau)=m|N(t)=k)\frac{\Pr(N(t)=k)}{\Pr(N(t+\tau)=m)} \\
&= \frac{\frac{(\lambda\tau)^{m-k}}{(m-k)!}e^{-\lambda\tau}\frac{(\lambda t)^k}{k!}e^{-\lambda t}}{\frac{(\lambda(t+\tau))^m}{m!}\exp(-\lambda(t+\tau))} \\
&= \binom{m}{k}\frac{t^k\tau^{m-k}}{(t+\tau)^m}.
\end{aligned}$$

Problem 8.27

$$\Pr(N(t)<10)=\sum_{k=0}^{9}\frac{(\lambda t)^k}{k!}e^{-\lambda t}$$

(a)

$$\lambda=0.1, t=10 \Rightarrow Pr(N(t)<10)=\sum_{k=0}^{9}\frac{(1)^k}{k!}e^{-1}\approx 1.$$

(b)

$$\lambda=10, t=10 \Rightarrow Pr(N(t)<10)=\sum_{k=0}^{9}\frac{(100)^k}{k!}e^{-100}\approx 0.$$

(c)

$$\begin{aligned}
\Pr(\text{1 call in 10 minutes}) &= 1\cdot e^{-1}=0.3679. \\
\Pr(\text{2 calls in 10 minutes}) &= \frac{1^2}{2!}\cdot e^{-1}=0.1839. \\
\Pr(\text{1 call, 2 calls}) &= \Pr(\text{1 call})\Pr(\text{2 calls}) \\
&= \frac{1^3}{2!1!}e^{-2}=0.0677.
\end{aligned}$$

Attention Please!

Problem 6.10 (Parts b and c are Revised!)

Assume the multivariate normal random variables $X=[x_1, x_2, \ldots, x_N]^T$ with mean vector of μ and covariance matrix of Σ. If we partition the X to two groups of $X_1=[x_1, x_2, \ldots, x_q]^T$ and $X_2=[x_{q+1}, x_{q+2}, \ldots, x_q]^T$, then we can write:

$$X = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \quad \text{with sizes} \quad \begin{bmatrix} q & 1 \\ (N-q) & 1 \end{bmatrix}$$

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \quad \text{with sizes} \quad \begin{bmatrix} q & 1 \\ (N-q) & 1 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \quad \text{with sizes} \quad \begin{bmatrix} q \quad q & q \quad (N-q) \\ (N-q) \quad q & (N-q) \quad (N-q) \end{bmatrix}$$

Then the distribution of X_1 conditioned on X_2=a is also multivariate normal $(X_1| X_2=a) \sim N(\mu_C,\Sigma_C)$, where

$$\mu_C = \mu_1 + \Sigma_{12} \Sigma_{22}^{-1} (a - \mu_2)$$
$$\Sigma_C = \Sigma_{11} - \Sigma_{12} \Sigma_{22}^{-1} \Sigma_{21}$$

a) Using the above information for our particular problem

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \ \mu = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \ \Sigma = \sigma^2 \begin{bmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{bmatrix}$$

We can define

$$\mu_1 = 0,\ \mu_2 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},\ a = \begin{bmatrix} x_2 \\ x_3 \end{bmatrix}$$

$$\Sigma_{11} = \sigma^2,\ \Sigma_{12} = \sigma^2 \left[\rho \quad \rho \right],\ \Sigma_{22} = \sigma^2 \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$

$$\begin{aligned} E\left[X_1 \mid X_2 = x_2, X_3 = x_3 \right] = \mu_{X_1 \mid X_2, X_3} &= \mu_1 + \Sigma_{12} \Sigma_{22}^{-1} \left(a - \mu_2 \right) \\ &= 0 + \sigma^2 \left[\rho \quad \rho \right] \frac{1}{\sigma^2 \left(1 - \rho^2 \right)} \begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix} \left(\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right) \\ &= \frac{\rho}{1+\rho} \left(x_2 + x_3 \right) \end{aligned}$$

b)

$$\begin{aligned} E\left[X_1 X_2 \mid X_3 = x_3 \right] &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x_1 x_2 f_{X_1, X_2 \mid X_3} \left(x_1, x_2 \right) dx_1 dx_2 \\ &= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x_1 x_2 f_{X_1 \mid X_2, X_3} \left(x_1 \right) f_{X_2 \mid X_3} \left(x_2 \right) dx_1 dx_2 \\ &= \int_{-\infty}^{+\infty} x_2 \left[\int_{-\infty}^{+\infty} x_1 f_{X_1 \mid X_2, X_3} \left(x_1 \right) dx_1 \right] f_{X_2 \mid X_3} \left(x_2 \right) dx_2 \\ &= \int_{-\infty}^{+\infty} x_2 E\left[X_1 \mid X_2 = x_3, X_3 = x_3 \right] f_{X_2 \mid X_3} \left(x_2 \right) dx_2 \\ &= E\left[x_2 E\left[X_1 \mid X_2 = x_3, X_3 = x_3 \right] \mid X_3 = x_3 \right] \\ &= E\left[x_2 \frac{\rho}{1+\rho} \left(x_2 + x_3 \right) \mid X_3 = x_3 \right] \\ &= \frac{\rho}{1+\rho} E\left[x_2^2 + x_2 x_3 \mid X_3 = x_3 \right] \\ &= \frac{\rho}{1+\rho} E\left[x_2^2 \mid X_3 = x_3 \right] + \frac{\rho}{1+\rho} E\left[x_2 x_3 \mid X_3 = x_3 \right] \end{aligned}$$

As $\mathrm{E}\left[\mathrm{g}\left(\mathrm{Y} \right) \mathrm{Z} \mid \mathrm{Y=y} \right] = \mathrm{g}\left(\mathrm{y} \right) \mathrm{E}\left[\mathrm{Z} \mid \mathrm{Y=y} \right]$, we can write

$$E\left[X_1 X_2 \mid X_3 = x_3 \right] = \frac{\rho}{1+\rho} E\left[x_2^2 \mid X_3 = x_3 \right] + \frac{\rho}{1+\rho} x_3 E\left[x_2 \mid X_3 = x_3 \right]$$

Again using the information provided in the previous page, we know that for a pair of jointly Gaussian random variables X_2 and X_3, the pdf of X_2 conditioned on X_3 would be a normal distribution by the following properties

$$X_2 : \left(\mu_2 + \rho_{X_2X3}\frac{\sigma_{X_2}}{\sigma_{X_3}}(x_3-\mu_3),\ \ \sigma_{X_2}^2\left(1-\rho_{X_2X3}^2\right) \right)$$

$$X_2 : \left(0+\rho\frac{\sigma}{\sigma}(x_3-0),\ \ \sigma^2\left(1-\rho^2\right) \right)$$

$$X_2 : \left(\rho x_3,\ \ \sigma^2\left(1-\rho^2\right)\right)$$

$$\rightarrow \begin{cases} E[X_2|X_3]=\rho x_3 \\ E[X_2^2|X_3]=\operatorname{var}[X_2|X_3]+\left(E[X_2|X_3]\right)^2=\sigma^2\left(1-\rho^2\right)+\left(\rho x_3\right)^2 \end{cases}$$

Thus

$$E\left[X_1X_2 \mid X_3=x_3\right]=\frac{\rho}{1+\rho}(\sigma^2\left(1-\rho^2\right)+\left(\rho x_3\right)^2+x_3\left(\rho x_3\right))$$
$$=\rho\sigma^2\left(1-\rho\right)+\left(\rho x_3\right)^2$$

c) Since $\mathrm{E}\left[\mathrm{g}(\mathrm{Y})Z\right]=\mathrm{E}\left[\mathrm{g}(\mathrm{Y})\,\mathrm{E}[\mathrm{Z}|\mathrm{Y}]\right]$, we can write

$$\begin{aligned} E\left[X_1X_2X_3\right] &= E\left[X_3E\left[X_1X_2 \mid X_3\right]\right] \\ &= E\left[X_3\left(\rho\sigma^2\left(1-\rho\right)+\left(\rho x_3\right)^2\right)\right] \\ &= E\left[X_3\sigma^2\rho\left(1-\rho\right)\right]+E\left[\rho^2X_3^3\right] \\ &= \sigma^2\rho\left(1-\rho\right)E\left[X_3\right]+\rho^2E\left[X_3^3\right] \\ &= 0 \end{aligned}$$

The last equality came from this fact that the all odd moments of a zero mean Gaussian distribution are zero.