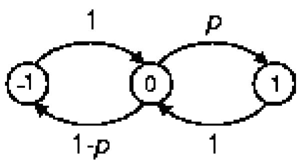
Solutions to chapter 9

Problem 9.5



a) The states of the system can be values of the voltage between switches. Hence there are three states, namely -1, 0, and +1. With this representation, the process is a random walk with reflecting boundaries. The Corresponding state transition matrix is

$$P = \begin{bmatrix} 0 & 1 & 0 \\ q & 0 & p \\ 0 & 1 & 0 \end{bmatrix}$$

b)

$$P^2 = \begin{bmatrix} q & 0 & p \\ 0 & 1 & 0 \\ q & 0 & p \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 0 & 1 & 0 \\ q & 0 & p \\ 0 & 1 & 0 \end{bmatrix}$$

$$P^4 = \begin{bmatrix} q & 0 & p \\ 0 & 1 & 0 \\ q & 0 & p \end{bmatrix}$$

We know that a *recurrent state* i is a *periodic state* if $p_{i,i}^{(n)} > 0$ for n>1. And also state i is recurrent if $P_{i,i}^{(n)} = 1$ for some $n \geq 1$. In above states, the only recurrent state is state 0 and two other states are transient. Therefore they can't be periodic and we have just to determine if state 0 is periodic and if so what its period is. For this process $p_{0,0}^{(n)} > 0$ only if n is even. Therefore, the process is periodic with period 2.

c) For $(i-1)t_s \leq t < it_s$ and i odd, x(t)=0. For $it_s \leq t < (i+1)t_s$ and i odd,

$$\Pr(X(t) = 1) = p$$
$$\Pr(X(t) = -1) = q = 1 - p$$

Problem 9.9

(a)

$$\mathbf{P} = \begin{bmatrix} 0.25 & 0.5 & 0.25 \\ 0.4 & 0.6 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \mathbf{Q}\Lambda\mathbf{Q}^{-1}.$$

Using MATLAB we found,

$$\mathbf{Q} = \begin{bmatrix} -0.4197 & 0.5774 & 0.2792 \\ 0.1570 & 0.5774 & -0.3981 \\ 0.8940 & 0.5774 & 0.8738 \end{bmatrix}, \qquad \Lambda = \begin{bmatrix} -0.4695 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0.3195 \end{bmatrix}.$$

The limiting form of the k-step transition probability matrix is then found as follows:

$$\lim_{k\to\infty} \mathbf{\Lambda}^k = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$\lim_{k\to\infty} \mathbf{P}^k = \mathbf{Q} \lim_{k\to\infty} \mathbf{\Lambda}^k \mathbf{Q}^{-1} = \begin{bmatrix} 0.4 & 0.5 & 0.1 \\ 0.4 & 0.5 & 0.1 \\ 0.4 & 0.5 & 0.1 \end{bmatrix}.$$

The steady-state distribution is then

$$\pi = \begin{bmatrix} 0.4 & 0.5 & 0.1 \end{bmatrix}.$$

(b) Using MATLAB to calculate $\mathbf{P}^{100}$, we get the same matrix found in part (a):

$$\mathbf{P}^{100} = \begin{bmatrix} 0.4 & 0.5 & 0.1 \\ 0.4 & 0.5 & 0.1 \\ 0.4 & 0.5 & 0.1 \end{bmatrix}.$$

$$\Rightarrow P_{1,3}^{(100)} = 0.1.$$

The interpretation of this result is that for all practical purposes, the process has reached steady state after 100 steps.

(c)

$$\pi(3) = \pi(0)\mathbf{P}^3 = \begin{bmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 0.25 & 0.5 & 0.25 \\ 0.4 & 0.6 & 0 \\ 1 & 0 & 0 \end{bmatrix}^3 = \begin{bmatrix} 0.4274 & 0.4808 & 0.0919 \end{bmatrix}$$

$$\Pr(\text{in state 3 after 3rd step}) = 0.0919.$$

Problem 9.11

(a)

$$
\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array}
\begin{array}{c}
\begin{array}{ccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{5}{12} & 0 & \frac{7}{12} & 0 & 0 & 0 & 0 \\
0 & \frac{5}{12} & 0 & \frac{7}{12} & 0 & 0 & 0 \\
0 & 0 & \frac{5}{12} & \frac{1}{6} & \frac{5}{12} & 0 & 0 \\
0 & 0 & 0 & \frac{7}{12} & 0 & \frac{5}{12} & 0 \\
0 & 0 & 0 & 0 & \frac{7}{12} & 0 & \frac{5}{12} \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\end{array}
$$

(b)

$$\Pr(\text{loses in 3 tosses}) = \Pr(3 \to 2 \to 1 \to 0) = (5/12)^3 = 0.0723.$$

(c)

$$\Pr(\text{loses in 4 tosses}) = \Pr(3 \to 3 \to 2 \to 1 \to 0) = \frac{1}{6}\left(\frac{5}{12}\right)^3 = 0.0121.$$

$$
\begin{aligned}
\Pr(\text{loses in 5 tosses}) &= \Pr(3 \to 3 \to 3 \to 2 \to 1 \to 0) \\
&+ \Pr(3 \to 4 \to 3 \to 2 \to 1 \to 0) \\
&+ \Pr(3 \to 2 \to 3 \to 2 \to 1 \to 0) \\
&+ \Pr(3 \to 2 \to 1 \to 2 \to 1 \to 0) \\
&= \left(\frac{1}{6}\right)^2 \left(\frac{5}{12}\right)^3 + 3\frac{7}{12}\left(\frac{5}{12}\right)^4 = 0.0548.
\end{aligned}
$$

$$\Pr(\text{loses in 5 or fewer tosses}) = 0.0723 + 0.0121 + 0.0548 = 0.1392.$$

We can verify this solution using MATLAB by noting that the probability of interest is found by finding the entry in the 4th row and 1st column of $\mathbf{P}^5$.

Solutions to supplementary exercises

Problem 1

For averaged periodograms or Bartlett's method we have:

$$\frac{\Delta f}{f_S} = \frac{0.89}{L} \qquad \text{Frequency Resolution}$$

$$\operatorname{var}\left[\hat{P}_B\left(e^{jw}\right)\right] \approx \frac{1}{K} P_X^2\left(e^{jw}\right) \qquad \text{Variance}$$

$$N = KL$$

Where L is length of samples per a time interval, K is number of time intervals and N is total number of samples. Therefore

$$\nu = \text{Variability=Relative Variance=}\frac{\operatorname{var}\left[\hat{P}_B\left(e^{jw}\right)\right]}{P_X^2\left(e^{jw}\right)} \approx \frac{1}{K}\text{=0.01} \rightarrow K = 100$$

$$\frac{\Delta f}{f_S} = \frac{0.89}{L} \rightarrow \frac{1}{1000} = \frac{0.89}{L} \rightarrow L = 890$$

$$N = KL = 89000$$

$$t = \frac{N}{f_S} = \frac{89000}{1000} = 89[s]$$

Problem 2

$$\nu = \frac{\operatorname{var}\left[\hat{P}_B\left(e^{jw}\right)\right]}{P_X^2\left(e^{jw}\right)} \approx \frac{1}{K}$$

$$\frac{\Delta f}{f_S} = \frac{0.89}{L} \rightarrow$$

$$\begin{cases} \dfrac{10}{10000} = \dfrac{0.89}{L_{Emilia}} \rightarrow L_{Emilia} = 890 \rightarrow t = \dfrac{N}{f_S} = \dfrac{N}{10000} = 10 \rightarrow N_{Emilia} = 10^5 \\ \dfrac{10}{100000} = \dfrac{0.89}{L_{Emil}} \rightarrow L_{Emil} = 8900 \rightarrow t = \dfrac{N}{f_S} = \dfrac{N}{100000} = 10 \rightarrow N_{Emil} = 10^6 \end{cases}$$

$$N = KL \rightarrow \begin{cases} K_{Emilia} = \dfrac{10^5}{890} = \dfrac{10^4}{89} \rightarrow \nu_{Emilia} = 0.0089 \\ K_{Emilia} = \dfrac{10^6}{8900} = \dfrac{10^4}{89} \rightarrow \nu_{Emil} = 0.0089 \end{cases}$$

As we can see there is no difference between normalized variances.

Problem 3

It is easy to recognize the underlying method is Blackman-Tukey method with following properties:

$$
\begin{aligned}
&\hat{P}_{BT}\left(e^{jw}\right)=\sum_{-M}^{M}\hat{r}_X\left(k\right)w\left(k\right)e^{-jw}\\
&\operatorname{var}\left[\hat{P}_{BT}\left(e^{jw}\right)\right]\approx P_X\left(e^{jw}\right)\frac{1}{N}\sum_{-M}^{M}w^2\left(k\right)\rightarrow \nu\approx\frac{1}{N}\sum_{-M}^{M}w^2\left(k\right)
\end{aligned}
$$

Therefore, we will have

$$
\begin{aligned}
&N=10000\\
&M=100\\
&w\left(n\right)=\begin{cases} e^{-0.1|n|} & ; \quad |n|\le 100\\ 0 & ; \quad |n|>100\end{cases}\\
&\nu\approx\frac{1}{N}\sum_{-M}^{M}w^2\left(k\right)=\frac{1}{10^4}\sum_{k=-100}^{100}e^{-0.2|k|}=\frac{1}{10^4}\left(2\sum_{k=0}^{100}e^{-0.2k}-1\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad=\frac{1}{10^4}\left(2\left(\frac{e^{-20.2}-1}{e^{-0.2}-1}\right)-1\right)\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad=10.0333*10^{-4}
\end{aligned}
$$

Problem 4

a)

$$x[n]+0.6x[n-1]-0.2x[n-2]=e[n]$$
$$x[n]=-0.6x[n-1]+0.2x[n-2]+e[n]$$
$$R[k]=E\left[x[n-k]\,x[n]\right]=E\left[x[n-k]\left(-0.6x[n-1]+0.2x[n-2]+e[n]\right)\right]$$
$$=-0.6R[k-1]+0.2R[k-2]+E\left[x[n-k]\,e[n]\right]$$
$$k=0: E\left[x[n-k]\,e[n]\right]=E\left[\left(-0.6x[n-1]+0.2x[n-2]+e[n]\right)e[n]\right]=1$$
$$k>0: E\left[x[n-k]\,e[n]\right]=0$$

$$k=0: R[0]=-0.6R[1]+0.2R[2]+1 \to R[0]+0.6R[1]-0.2R[2]=1$$
$$k=1: R[1]=-0.6R[0]+0.2R[1] \to 0.6R[0]-0.8R[1]=0$$
$$k=2: R[2]=-0.6R[1]+0.2R[0] \to -0.2R[0]+0.6R[1]+R[2]=0$$
$$|k|>2: R[k]=-0.6R[k-1]+0.2R[k-2]$$

$$\begin{bmatrix} 1 & 0.6 & -0.2 \\ 0.6 & -0.8 & 0 \\ 0.2 & 0.6 & 1 \end{bmatrix}\begin{bmatrix} R[0] \\ R[1] \\ R[2] \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} R[0] \\ R[1] \\ R[2] \end{bmatrix}=\begin{bmatrix} 2.4 \\ -1.8 \\ 1.6 \end{bmatrix}$$

Attention Please! $P[output]=H[z]H^*[\frac{1}{z^*}]P[input]$

$$H[z]=\frac{1}{1+0.6z^{-1}-0.2z^{-2}}$$
$$P_X[z]=H[z]H^*[\frac{1}{z^*}]P_E[z]$$
$$=\left(\frac{1}{1+0.6z^{-1}-0.2z^{-2}}\right)\left(\frac{1}{1+0.6z-0.2z^2}\right)1$$
$$=\frac{1}{1.4-0.6(z+z^{-1})-0.2(z^2+z^{-2})}$$
$$P_X(e^{j\omega})=\frac{1}{1.4-1.2\cos(\omega)-0.4\cos(2\omega)}$$

b)

$$x[n] = e[n] + 0.8e[n-1] + 0.2e[n-2]$$

$$\begin{aligned} R[k] &= E\left[x[n-k]\,x[n]\right] = E\left[\left(e[n-k] + 0.8e[n-k-1] + 0.2e[n-k-2]\right)\left(e[n] + 0.8e[n-1] + 0.2e[n-2]\right)\right] \\ &= 0.2R_{EE}(k+2) + 0.96R_{EE}(k+1) + 1.68R_{EE}(k) + 0.96R_{EE}(k-1) + 0.2R_{EE}(k-2) \end{aligned}$$

$k = 0 : R[0] = 1.68R_{EE}[0] = 1.68$

$k = 1 : R[1] = R[-1] = 0.96R_{EE}[0] = 0.96$

$k = 2 : R[2] = R[-2] = 0.4R_{EE}[0] = 0.2$

$|k| > 2 : R[k] = 0$

$$\begin{aligned} P_X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} R_{XX}[n]e^{-j\omega n} = \sum_{n=-2}^{2} R_{XX}[n]e^{-j\omega n} &= 1.68 + 0.96\left(e^{j\omega} + e^{-j\omega}\right) + 0.2\left(e^{2j\omega} + e^{-2j\omega}\right) \\ &= 1.68 + 1.92\cos(\omega) + 0.2\cos(2\omega) \end{aligned}$$

or we could simply write

$$\begin{aligned} & x[n] = e[n] + 0.8e[n-1] + 0.2e[n-2] \\ & \rightarrow X[z] = E[z](1 + 0.8z^{-1} + 0.2z^{-2}) \\ & \rightarrow H[z] = 1 + 0.8z^{-1} + 0.2z^{-2} \\ & P_X[z] = H[z]H^*[\frac{1}{z^*}]P_E[z] \\ & \qquad = \left(1 + 0.8z^{-1} + 0.2z^{-2}\right)\left(1 + 0.8z^{1} + 0.2z^{2}\right)1 \\ & \qquad = 1.68 + 0.96(z + z^{-1}) + 0.2(z^2 + z^{-2}) \\ & P_X(e^{j\omega}) = 1.68 + 1.92\cos(\omega) + 0.4\cos(2\omega) \end{aligned}$$

Problem 6

$$x[n]+a_1x[n-1]+a_2x[n-2]=e[n]$$

Multiplying by $x[n-k];\ k\geq 0$ and taking the expectation leads to the relation

$$r[k]+a_1r[k-1]+a_2r[n-2]=E[x[n-k]e[n]]$$

$$\begin{matrix} k=0: \\ k=1: \\ k=2: \\ k=3: \\ k=4: \end{matrix} \begin{bmatrix} r[0] & r[1] & r[2] \\ r[1] & r[0] & r[1] \\ r[2] & r[1] & r[0] \\ r[3] & r[2] & r[1] \\ r[4] & r[3] & r[2] \end{bmatrix} \begin{bmatrix} 1 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \sigma_e^2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 7.73 & 6.8 & 4.75 \\ 6.8 & 7.73 & 6.8 \\ 4.75 & 6.8 & 7.73 \\ 2.36 & 4.75 & 6.8 \\ 0.23 & 2.36 & 4.75 \end{bmatrix}}_{\mathbf{A}} \underbrace{\begin{bmatrix} 1 \\ a_1 \\ a_2 \end{bmatrix}}_{\mathbf{x}} = \underbrace{\begin{bmatrix} \sigma_e^2 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{b}}$$

The above equation which has the form of Ax=b, can be solved by

$$\mathbf{x}=\left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{b}$$

$$\rightarrow \begin{bmatrix} 1 \\ a_1 \\ a_2 \end{bmatrix} = \sigma_e^2 \begin{bmatrix} 1.1061 \\ -1.6451 \\ 0.7655 \end{bmatrix} \rightarrow \begin{cases} \sigma_e^2 \approx 0.9041 \\ a_1 \approx -1.4873 \\ a_2 \approx 0.6921 \end{cases}$$

Problem 8

$$d[n]-0.8d[n-1]=e[n]$$

Multiplying by $d[n-k];\ k\geq 0$ and taking the expectation leads to the relation

$$r[k]-0.8r[k-1]=E[d[n-k]e[n]]$$

$$\begin{array}{l} k=0: \\ k=1: \end{array}\begin{bmatrix} 1 & -0.8 \\ -0.8 & 1 \end{bmatrix}\begin{bmatrix} r[0] \\ r[1] \end{bmatrix}=\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$\rightarrow \begin{cases} r[0]=2.7778 \\ r[k]=0.8r[k-1];\ k\geq 1 \end{cases}$$

Also we have

$$D[z](1+0.5z^{-1})+V[z]=X[z]\rightarrow d[n]+0.5d[n-1]+v[n]=x[n]$$

If we estimate $\hat{d}[n]=w_0x[n]+w_1x[n-1]$, then the squared error can be estimated as follows

$$\begin{aligned} E[e^2]&=E[(d[n]-\hat{d}[n])^2] \\ &=E\left[\left(d[n]-w_0\left(d[n]+0.5d[n-1]+v[n]\right)+w_1\left(d[n-1]+0.5d[n-2]+v[n-1]\right)\right)^2\right] \\ &=w_0^2\left(1.25r[0]+r[1]+0.1\right)+w_1^2\left(1.25r[0]+r[1]+0.1\right)+w_0\left((w_1-2)r[0]+(2.5w_1-1)r[1]+w_1r[2]\right) \\ &+w_1\left(-2r[1]-r[2]\right)+r[0] \\ &=5.79445w_0^2+5.79445w_1^2+10.1111w_0w_1-7.7778w_0-6.2222w_1+2.7778 \end{aligned}$$

Taking the derivative of $E[e^2]$ according to w_0 and w_1, we get

$$\frac{\partial E[e^2]}{\partial w_0} = 11.5889 w_0 + 10.1111 w_1 - 7.7778 = 0$$

$$\frac{\partial E[e^2]}{\partial w_1} = 11.5889 w_1 + 10.1111 w_0 - 6.2222 = 0$$

$$\rightarrow \begin{bmatrix} 11.5889 & 10.1111 \\ 10.1111 & 11.5889 \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = \begin{bmatrix} 7.7778 \\ 6.2222 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = \begin{bmatrix} 0.8489 \\ -0.2037 \end{bmatrix}$$