MVE135 Random Processes with Applications Fall 2010

Written Exam Thursday 21 October 2.00 pm - 6.00 pm

TEACHER AND JOUR: Patrik Albin and Krzysztof Bartoszek, respectively.

AIDS: Beta.

GRADES: 12 (40%), 18 (60%) and 24 (80%) points for grade 3, 4 and 5, respectively.

Task 1. Calculate $\Pr(3 < X \leq 4 | 2 < X \leq 4)$ for a continuous random variable X with PDF $f_X(x) = 2\,\mathrm{e}^{-2x}$ for $x \geq 0$ and $f_X(x) = 0$ for $x < 0$. **(5 points)**

Task 2. Let X and Y be independent continuous random variables that are both Gaussian distributed with zero mean and unit variance 1. Show that the random variable $Z = X^2 + Y^2$ has PDF $f_Z(z) = \frac{1}{2}\,\mathrm{e}^{-z/2}$ for $z \geq 0$ and $f_Z(z) = 0$ for $z < 0$. (Students who have a working plan for how the proof can be done, but are not able to carry out all steps of the plan in full detail will get awarded for describing their plan.) **(5 points)**

Task 3. Express in terms of the Φ-function $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-y^2/2}\,dy$ the probability $\Pr(2X(1) - X(2) \leq 3)$ for a WSS Gaussian process $X(t)$, $t \in \mathbb{R}$, with mean $\mu_X = 1$ and autocorrelation function $R_{XX}(\tau) = \mathrm{e}^{-|\tau|} + 1$ for $\tau \in \mathbb{R}$. **(5 points)**

Task 4. Calculate the autocovariance function $C_{XX}(s,t)$ for a unit rate ($\lambda = 1$) Poisson process $X(t)$, $t \geq 0$. (Of course, students may make use of the fact that Poisson processes have Poisson distributed independent increments.) **(5 points)**

Task 5. Calculate the crosscorrelation $R_{XY}(0)$ for two WSS processes $X(t)$ and $Y(t)$, $t \in \mathbb{R}$, with cross spectral density $S_{XY}(f) = 2/(1 + (2\pi f)^2)$. **(5 points)**

Task 6. Let $\ldots, X(-1), X(0), X(1), \ldots$ be zero-mean uncorrelated random variables with unit variance 1. Find the autocorrelation function $R_{YY}(\ell, \ell + k)$ for the process $Y(n) = X(n) - \frac{1}{2}X(n-1)$ for $n \in \mathbb{Z}$. **(5 points)**

$\boxed{\textbf{Good Luck!}}$

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Solutions to Written Exam Thursday 21 October

Task 1. $\Pr(3<X\leq 4|2<X\leq 4) = \Pr(3<X\leq 4, 2<X\leq 4)/\Pr(2<X\leq 4) = \Pr(3<X\leq 4)/\Pr(2<X\leq 4) = \int_3^4 f_X(x)\,dx/\int_2^4 f_X(x)\,dx = (\mathrm{e}^{-6}-\mathrm{e}^{-8})/(\mathrm{e}^{-4}-\mathrm{e}^{-8})$.

Task 2. $\mathbf{E}\{\mathrm{e}^{j\omega(X^2+Y^2)}\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{j\omega(x^2+y^2)}f_X(x)f_Y(y)\,dxdy = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\mathrm{e}^{j\omega(x^2+y^2)}\frac{1}{2\pi}\,\mathrm{e}^{-(x^2+y^2)/2}\,dxdy = \int_{r=0}^{r=\infty}\int_{\theta=0}^{\theta=2\pi}\frac{1}{2\pi}\,\mathrm{e}^{-(1-2j\omega)r^2/2}\,r\,d\theta dr = 1/(1-2j\omega) = \int_0^{\infty}\mathrm{e}^{j\omega z}\frac{1}{2}\,\mathrm{e}^{-z/2}\,dz$.

Task 3. $\Pr(2X(1)-X(2)\leq 3) = \Phi((3-\mu)/\sigma)$, where $\mu = \mathbf{E}(2X(1)-X(2)) = 2\mu_X-\mu_X = 1$ and $\sigma^2 = \mathbf{Var}(2X(1)-X(2)) = 4C_{XX}(0)-4C_{XX}(1)+C_{XX}(0) = 5(R_{XX}(0)-\mu_X^2) - 4(R_{XX}(1)-\mu_X^2) = 5-4\,\mathrm{e}^{-1}$.

Task 4. $C_{XX}(s,t) = \mathbf{Cov}(X(s),X(t)) = \mathbf{Cov}(X(\min\{s,t\}),X(\max\{s,t\})) = \mathbf{Cov}(X(\min\{s,t\}),X(\min\{s,t\}))+\mathbf{Cov}(X(\min\{s,t\}),X(\max\{s,t\})-X(\min\{s,t\})) = \mathbf{Var}(X(\min\{s,t\}))+0 = \mathbf{Var}(\mathrm{Po}(\min\{s,t\})) = \ldots = \min\{s,t\}$.

Task 5. $R_{XY}(0) = F^{-1}(S_{XY})(0) = \int_{-\infty}^{\infty}\mathrm{e}^{j2\pi f\tau}S_{XY}(f)\,df\big|_{\tau=0} = \int_{-\infty}^{\infty}S_{XY}(f)\,df = \int_{-\infty}^{\infty}2/(1+(2\pi f)^2)\,df = 1$.

Task 6. $S_{YY}(f) = F(R_{YY})(f) = \sum_{k=-\infty}^{\infty}\mathrm{e}^{-j2\pi fk}R_{YY}(\ell,\ell+k) = |H(f)|^2S_{XX}(f)$, where $H(f) = 1-\frac{1}{2}\mathrm{e}^{-j2\pi f}$ and $S_{XX}(f)=1$, giving $S_{YY}(f) = |1-\frac{1}{2}\mathrm{e}^{-j2\pi f}|^2 = \frac{5}{4}-\frac{1}{2}\mathrm{e}^{j2\pi f}-\frac{1}{2}\mathrm{e}^{-j2\pi f}$, so that $R_{YY}(\ell,\ell) = \frac{5}{4}$ and $R_{YY}(\ell,\ell\pm 1) = -\frac{1}{2}$, while $R_{YY}(\ell,\ell+k) = 0$ for $k>1$, for $\ell\in\mathbb{Z}$.