MVE136 Random Signals Analysis

Written exam Monday 20 August 2012 8.30 am – 12.30 pm

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AIDS: Beta or 2 sheets (4 pages) of hand-written notes (computer print-outs and/or xerox-copies are not allowed), but not both these aids.

GRADES: 12, 18 and 24 points for grades 3, 4 and 5, respectively. GOOD LUCK!

Task 1. A pair of nonnegative integer valued random variables (X,Y) have joint probability generating function $H_{X,Y}(z_1,z_2) = \mathbf{E}\{z_1^X z_2^Y\}$ given by

$$H_{X,Y}(z_1,z_2) = \frac{1}{1+a\,(1-z_1)+b\,(1-z_2)} \quad \text{for } 0 \le z_1, z_2 \le 1,$$

where $a, b > 0$ are real constants. Find $\mathbf{E}\{X^2 Y\}$. **(5 points)**

Task 2. Show by example that the random process $Z(t) = X(t) + Y(t)$ may be a wide sense stationary process even though the random processes $X(t)$ and $Y(t)$ are not.

(5 points)

Task 3. Consider a discrete time Markov chain $X(n)$ with state space E, initial distribution $\pi(0)$ and transition probability matrix P given by

$$E = \{0,1\}, \quad \pi(0) = [1/2 \;\; 1/2] \quad \text{and} \quad P = \begin{bmatrix} 2/3 & 1/3 \\ 1/3 & 2/3 \end{bmatrix},$$

respectively. Find the probability $\mathbf{P}\{X(5)=1 \,|\, X(2)=1\}$. **(5 points)**

Task 4. Let $X(t)$ be a continuous-time random process with power spectral density $S_{XX}(f)$. The derivative process of $X(t)$ is defined as $X'(t) = \lim_{h\to 0}(X(t+h)-X(t))/h$ whenever this limit is well-defined in a suitable sense. Show that the cross spectral density between $X(t)$ and $X'(t)$ is given by $S_{XX'}(f) = j2\pi f\, S_{XX}(f)$. **(5 points)**

Task 5. Let a pair of zero-mean jointly Gaussian continuous time processes $X_1(t)$ and $X_2(t)$ be inputs to linear filters with impulse responses $h_1(t)$ and $h_2(t)$, respectively, and corresponding outputs $Y_1(t)$ and $Y_2(t)$. Under what exact (i.e., necessary and sufficient) conditions on the crosscorrelation function $R_{X_1X_2}(t_1,t_2) = \mathbf{E}\{X_1(t_1)X_2(t_2)\}$ are two output values $Y_1(s)$ and $Y_2(t)$ independent? **(5 points)**

Task 6. Suppose $x[n]$ is an AR(1) process with $a_1 = 0.5$. Derive and illustrate (plot) the power spectral density of $x[n]$, assuming that the white noise input has variance 2.

(5 points)

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Solutions to written exam Monday 20 August 2012

Task 1. Proceeding as in Example 5.19 in the book of Miller and Childers we find that $\mathbf{E}\{X^2Y\} = 6\,a^2b + 2\,a\,b$.

Task 2. This is one of the home exercises listed for the course. For example, given any wide sense stationary process $Z(t)$ we may take $X(t) = Z(t)/2-t$ and $Y(t) = Z(t)/2+t$.

Task 3. We have $\mathbf{P}\{X(5)=1\,|\,X(2)=1\} = (P^3)_{1,1}$ (i.e., the lower diagonal element of the third power of P), which by elementary matrix calculations equals 14/27.

Task 4. As $R_{XX'}(\tau) = \mathbf{E}\big\{X(t)\lim_{h\to 0}(X(t+\tau+h)-X(t+\tau))/h\big\} = \lim_{h\to 0}\mathbf{E}\big\{X(t)(X(t+\tau+h)-X(t+\tau))/h\big\} = \lim_{h\to 0}(R_{XX}(\tau+h)-R_{XX}(\tau))/h = R'_{XX}(\tau)$, we have $S_{XX'}(f) = \int_{-\infty}^{\infty} R_{XX'}(\tau)\,\mathrm{e}^{-j2\pi f\tau}\,d\tau = \int_{-\infty}^{\infty} R'_{XX}(\tau)\,\mathrm{e}^{-j2\pi f\tau}\,d\tau = j2\pi f\int_{-\infty}^{\infty} R_{XX}(\tau)\,\mathrm{e}^{-j2\pi f\tau}\,d\tau = j2\pi f\,S_{XX}(f)$.

Task 5. As the outputs $Y_1(s)$ and $Y_2(t)$ are jointly Gaussian, the exact condition is that their crosscorrelation is zero, which is to say that $R_{Y_1Y_2}(s,t) = \mathbf{E}\big\{\big(\int_{-\infty}^{\infty} h_1(u)X_1(s-u)\,du\big)\big(\int_{-\infty}^{\infty} h_2(v)X_2(t-v)\,dv\big)\big\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(u)h_2(v)\,\mathbf{E}\{X_1(s-u)X_2(t-v)\}\,du dv = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_1(u)h_2(v)\,R_{X_1X_2}(s-u,t-v)\,du dv = 0$.

Task 6. It is given that

$$x[n] + 0.5x[n-1] = e[n],$$

where $e[n]$ is a white noise process such that $E\{e[n]^2\} = 2$. An equivalent description is that $x[n]$ is the output from a linear system, with the transfer function

$$H(z) = \frac{1}{1+0.5z^{-1}},$$

where $e[n]$ is the input signal. The PSD of $x[n]$ is therefore

$$\begin{aligned} P_x\left(e^{j\omega}\right) &= \left|H\left(e^{j\omega}\right)\right|^2 P_e\left(e^{j\omega}\right) \\ &= \frac{\sigma_e^2}{\left|1+0.5e^{-j\omega}\right|^2} = \frac{2}{\left|1+0.5e^{-j\omega}\right|^2}. \end{aligned}$$

In order to plot (sketch) $P_x\left(e^{j\omega}\right)$ it helps to notice that $H(z)$ has a pole in $z = -0.5$, since this tells us that it is a high-pass filter. Further, it is easy to see that $P_x\left(e^{j\pi}\right) = 8$. A detailed plot of PSD is given in Figure 1 below.

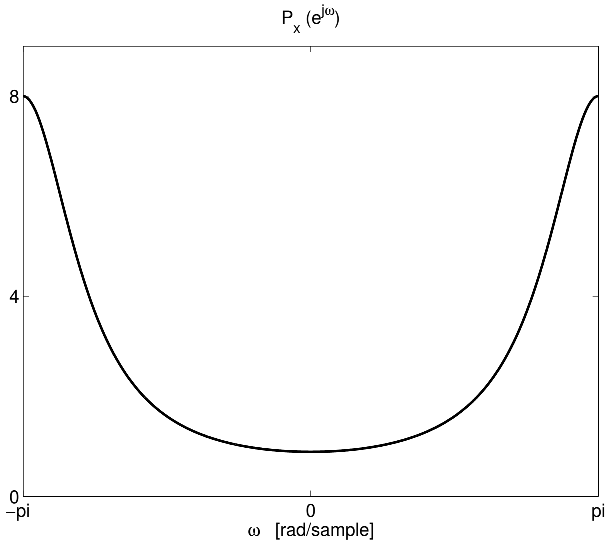

Figure 1: The power spectral density of $x[n]$.