MVE136 Random Signals Analysis Fall 2012

Written exam Thursday 25 October 2012 2.00 – 6.00 pm

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AIDS: Beta <u>or</u> 2 sheets (4 pages) of hand-written notes (computer print-outs and/or xerox-copies are not allowed), but not both these aids.

GRADES: 12 (40%), 18 (60%) and 24 (80%) points for grade 3, 4 and 5, respectively.

GOOD LUCK!

Task 1. Find the conditional expectation $E(X\,|\,Y=\ell)$ for a pair of discrete random variables X and Y with PMF $P_{X,Y}(k,\ell) = (k!\,\ell!\,\mathrm{e}^2)^{-1}$ for $k,\ell \in \{0,1,2,\ldots\}$. **(5 points)**

Task 2. Find the probability $Pr(X(1)+X(2)+X(3)>6)$ for a continuous time WSS Gaussian process $X(t)$ with mean $\mu_X = 1$ and autocorrelation function $R_{XX}(\tau) = \mathrm{e}^{-|\tau|}+1$ for $\tau \in \mathbb{R}$. **(5 points)**

Task 3. Consider a Markov chain $X[k] \in \{0,1\}$ with initial distribution $\pi(0)$ and transition matrix P given by

$$\pi(0) = (1/2, 1/2) = [1/2 \;\; 1/2] \quad \text{and} \quad P = \begin{bmatrix} 0 & 1 \\ 2/3 & 1/3 \end{bmatrix},$$

respectively. [The two different representations of $\pi(0)$ are in vector form (first) and row matrix form (secondly): It is a matter of taste which one one prefers – both are correct.] Find the expected value $E[X[2]]$ of $X[2]$. **(5 points)**

Task 4. Calculate the autocorrelation estimate $\hat{R}^{(\mathrm{tri})}_{XX}(\tau) = \frac{1}{2t_0}\int_{-t_0+|\tau|/2}^{t_0-|\tau|/2} X(t-\frac{\tau}{2})X(t+\frac{\tau}{2})\,dt$ for $\tau \in (-2t_0, 2t_0)$ if the observed signal $X(t)$ over the interval $(-t_0,t_0)$ is given by $\cos(t)$ for $t \in (-t_0,t_0)$. [Hint: The trigonometric formula $\cos(x+y)+\cos(x-y) = 2\cos(x)\cos(y)$ might become useful.] Say something about how the corresponding PSD estimate $\hat{S}^{(\mathrm{tri})}_{XX}(f)$ is calculated (but you need not do all details of the calculation). **(5 points)**

Task 5. Let continuous time white noise $N(t)$ [a zero-mean WSS process with PSD $S_{NN}(f) = N_0/2$ for $f \in \mathbb{R}$] be input to an LTI system with impulse response $h(t) = 2/(1+(2\pi t)^2)$ for $t \in \mathbb{R}$. Find the average power $E[Y(t)^2]$ of the output $Y(t)$ from the LTI system. **(5 points)**

Task 6. Compute the autocorrelation function $r_x[n]$ for $n = 0$ and $n = 1$ when $x[n]$ is an AR(1)-process with parameter $a_1 = 0.7$. You can assume that the input noise has variance $\sigma_e^2 = 1$. **(5 points)**

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Solutions to written exam Thursday 25 October 2012

Task 1. As $P_Y(\ell)=\sum_{k=0}^{\infty} P_{X,Y}(k,\ell)=(\ell!\,\mathrm{e}^1)^{-1}$ we get $P_{X\,|\,Y}(k\,|\,\ell)=P_{X,Y}(k,\ell)/P_Y(\ell)$ $=(k!\,\mathrm{e}^1)^{-1}$ so that $E(X\,|\,Y=\ell)=\sum_{k=0}^{\infty} k\,P_{X\,|\,Y}(k\,|\,\ell)=\sum_{k=0}^{\infty} k/(k!\,\mathrm{e}^1)=E[\mathrm{Po}(1)]=1$.

Task 2. As $X(1)+X(2)+X(3)$ is $N(m,\sigma^2)$-distributed we have $Pr(X(1)+X(2)+X(3)>6)=Pr(N(m,\sigma^2)>6)=1-\Phi((6-m)/\sigma)$, where $m=E[X(1)+X(2)+X(3)]=3$ and $\sigma^2=\mathrm{Var}(X(1)+X(2)+X(3))=3\,C_{XX}(0)+4\,C_{XX}(1)+2\,C_{XX}(2)=3+4\,\mathrm{e}^{-1}+2\,\mathrm{e}^{-2}$ [using that $C_{XX}(\tau)=R_{XX}(\tau)-\mu_X^2=\mathrm{e}^{-|\tau|}$].

Task 3. As

$$\pi(2)=\pi(0)\,P^2=[1/2\ \ 1/2]\begin{bmatrix}0 & 1\\ 2/3 & 1/3\end{bmatrix}^2=[1/2\ \ 1/2]\begin{bmatrix}2/3 & 1/3\\ 2/9 & 7/9\end{bmatrix}=[4/9\ \ 5/9],$$

we have $E[X[2]]=\sum_{k=0}^{1} k\,P_{X[2]}(k)=\sum_{k=0}^{1} k\,\pi(2)_k=0\cdot(4/9)+1\cdot(5/9)=5/9$.

Task 4. We have $\hat{R}_{XX}^{(\mathrm{tri})}(\tau)=\frac{1}{2t_0}\int_{-t_0+|\tau|/2}^{t_0-|\tau|/2} X(t-\frac{\tau}{2})X(t+\frac{\tau}{2})\,dt=\frac{1}{2t_0}\int_{-t_0+|\tau|/2}^{t_0-|\tau|/2}\cos(t-\frac{\tau}{2})\cos(t+\frac{\tau}{2})\,dt=\frac{1}{4t_0}\int_{-t_0+|\tau|/2}^{t_0-|\tau|/2}(\cos(\tau)+\cos(2t))\,dt=\frac{2t_0-|\tau|}{4t_0}\cos(\tau)+\frac{1}{4t_0}\sin(2t_0-|\tau|)$ for $\tau\in(-2t_0,2t_0)$. This gives $\hat{S}_{XX}^{(\mathrm{tri})}(f)=(F\hat{R}_{XX}^{(\mathrm{tri})})(f)=\frac{1}{4t_0}\int_{-2t_0}^{2t_0}\mathrm{e}^{-j2\pi f\tau}\big((2t_0-\tau|)\cos(\tau)+\sin(2t_0-|\tau|)\big)\,d\tau=\ldots$.

Task 5. As $H(f)=(Fh)(f)=\mathrm{e}^{-|f|}$ we have $E[Y(t)^2]=\int_{-\infty}^{\infty} S_{YY}(f)\,df=\int_{-\infty}^{\infty}|H(f)|^2\,S_{NN}(f)\,df=(N_0/2)\int_{-\infty}^{\infty}\mathrm{e}^{-2|f|}\,df=N_0/2$.

Task 6. We can use the Yule-Walker (YW) equations to find $r_x[0]$ and $r_x[1]$. Since it is simple, this solution will start with a derivation of the YW-equations: If we multiply both sides of the equation

$$x[n]+0.7x[n-1]=e[n]$$

with $x[n-k]$ and take expectations we get

$$\underbrace{E\{x[n-k](x[n]+0.7x[n-1])\}}_{r_x[k]+0.7r_x[k-1]}=\underbrace{E\{x[n-k]e[n]\}}_{\delta[k]}\quad \text{for } k\geq 0.$$

Using this equation for $k=0$ and $k=1$ we get the matrix equation

$$\begin{bmatrix}1 & 0.7\\ 0.7 & 1\end{bmatrix}\begin{bmatrix}r_x[0]\\ r_x[1]\end{bmatrix}=\begin{bmatrix}1\\ 0\end{bmatrix}.$$

Solving this matrix equation we get $r_x[0]=1/0.51\approx 1.96$ and $r_x[1]=-0.7r_x[0]\approx -1.37$.