MVE136 Random Signals Analysis

Written exam Wednesday 23 October 2013 2 – 6 pm

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AIDS: Beta or 2 sheets (4 pages) of hand-written notes (computer print-outs and/or xerox-copies are not allowed), but not both these aids.

GRADES: 12 (40%), 18 (60%) and 24 (80%) points for grade 3, 4 and 5, respectively.

GOOD LUCK!

Task 1. Two jointly Gaussian random variables X and Y have PDF given by

$$f_{X,Y}(x,y) = \frac{9}{\pi\sqrt{8}} \exp\left\{-\frac{36x^2 - 36xy + 81y^2}{16}\right\} \quad \text{for } x, y \in \mathbb{R}.$$

Indentify the variances σ_X^2 and σ_Y^2 of X and Y, respectively, as well as their correlation coefficient $\rho_{X,Y}$. **(5 points)**

Task 2. Let $X(t)$ be a continuous time WSS Gaussian random process with mean $\mu_X = 1$ and autocovariance function $C_{X,X}(\tau) = \mathrm{e}^{-|\tau|}$ for $\tau \in \mathbb{R}$. Form a new process $Y(t)$ as $Y(t) = X(t)\cos(\omega t + \theta)$ for $t \in \mathbb{R}$, where ω and θ are real constants. Is $Y(t)$ WSS? Is $Y(t)$ Gaussian? **(5 points)**

Task 3. A certain workstation has a life length until failure that is an exponentially distributed random variable with mean 250 days. Assume that you have just purchased 10 new such workstations that each come with a 90-day warranty. What is the probability that at least one of these 10 workstations fails before the end of the warranty period? **(5 points)**

Task 4. For a Markov chain with each of the transition probability matrices (a)-(c) given below, find the states that communicate with each other together with the periodicities of these various states.

$$\text{(a)} \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 \end{bmatrix} \quad \text{(b)} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1/3 & 0 & 2/3 & 0 \end{bmatrix} \quad \text{(c)} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1 \\ 0 & 1/2 & 1/2 & 0 \end{bmatrix} \quad \textbf{(5 points)}$$

Task 5. The sum $X(t) = S(t) + N(t)$ of a continuous time signal process $S(t)$ with PSD $S_{SS}(f) = 2/(2 + (2\pi f)^2)$ and an independent (of $S(t)$) continuous time noise process $N(t)$ with PSD $S_{NN}(f) = 1$ is input to an LTI filter. Determine the impulse response of the filter that produces an output $Y(t)$ from the filter that minimizes the mean-square signal-recovering error $E[(S(t)-Y(t))^2]$. Also, find the PSD of the output $Y(t)$ when the filter has this impulse response. **(5 points)**

Task 6. Consider the ARMA$(1,1)$ process

$$x[n]-0.8\,x[n-1]=e[n]-0.9\,e[n-1],$$

where $e[n]$ is a wide sense stationary (WSS) white (noise) process with variance $\sigma_e^2 = 1$. Compute and sketch the power spectral density (PSD) of the ARMA(1,1) process $x[n]$. **(5 points)**

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Solutions to written exam Wednesday 23 October 2013

Task 1. In the general formula for a jointly Gaussian PDF

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho_{X,Y}^2}}\exp\left\{-\frac{\left(\frac{x-\mu_X}{\sigma_X}\right)^2 - 2\rho_{X,Y}\left(\frac{x-\mu_X}{\sigma_X}\right)\left(\frac{y-\mu_Y}{\sigma_Y}\right) + \left(\frac{y-\mu_Y}{\sigma_Y}\right)^2}{2(1-\rho_{X,Y}^2)}\right\}$$

we see that we have $\mu_X = \mu_Y = 0$ as non-zero μ_X and/or μ_Y only generate multiples of x and y together with constants in the numerator of the exponent which do not feature in the PDF given in the task. Further we see by comparison that we must have

$$\frac{(y/\sigma_Y)^2}{(x/\sigma_X)^2} = \frac{81\,y^2}{36\,x^2}, \quad \frac{2\rho_{X,Y}(x/\sigma_X)(y/\sigma_Y)}{(x/\sigma_X)^2} = \frac{y}{x} \quad \text{and} \quad \frac{(x/\sigma_X)^2}{2(1-\rho_{X,Y}^2)} = \frac{36\,x^2}{16},$$

which gives $\sigma_X = 1/2$, $\sigma_Y = 1/3$ and $\rho_{X,Y} = 1/3$.

Task 2. The mean function $\mu_Y(t) = E[X(t)\cos(\omega t+\theta)] = \mu_X(t)\,\cos(\omega t+\theta) = \cos(\omega t+\theta)$ of $Y(t)$ depends on t so that $Y(t)$ is not WSS. However, $Y(t)$ is Gaussian as a linear combination of $Y(t)$-process values at different times $\sum_{i=1}^n a_i Y(t_i) = \sum_{i=1}^n a_i \cos(\omega t_i + \theta)X(t_i)$ is also a linear combination of $X(t)$-process values at different times and therefore normal distributed (since $X(t)$ is Gaussian).

Task 3. The probability that a single workstation fails before the end of the warranty is $\int_0^{90} \frac{1}{250}\,\mathrm{e}^{-x/250}\,dx = 1-\mathrm{e}^{-9/25}$. Therefore the probability that at least one of the 10 workstations fails before the end of the warranty is $1-(1-(1-\mathrm{e}^{-9/25}))^{10} = 1-\mathrm{e}^{-18/5}$.

Task 4. By inspection, for chain (a) the first three states communicate with common period 1, for chain (b) all four states communicate with common period 3, while for chain (c) all four states communicate with common period 2.

Task 5. We are looking for the impulse response of the Wiener filter with transfer function $H(f) = S_{SS}(f)/(S_{SS}(f)+S_{NN}(f)) = 2/(4+(2\pi f)^2)$ so that the impulse response (see Table E.1 in the book of Miller and Childers) is $h(t) = 2\,\mathrm{e}^{-2|t|}$. Further, we have

$$S_{YY}(f) = |H(f)|^2 S_{XX}(f) = \frac{S_{SS}(f)^2}{S_{SS}(f)+S_{NN}(f)} = \frac{4}{(4+(2\pi f)^2)\,(2+(2\pi f)^2)}.$$

Task 6. The difference equation described in the problem formulation corresponds to a linear filter with the transfer function

$$H(z) = \frac{1-0.9\,z^{-1}}{1-0.8\,z^{-1}}.$$

The power spectral density of the output signal is therefore

$$P_x(e^{j\omega}) = |H(e^{j\omega})|^2\sigma_e^2 = \frac{|1-0.9\,e^{-j\omega}|^2}{|1-0.8\,e^{-j\omega}|^2}.$$

In order to sketch the power spectral density it is helpful to note that $H(z)$ has a pole in $z = p_1 = 0.8$ and a zero in $z = z_1 = 0.9$, see Figure 1 below.

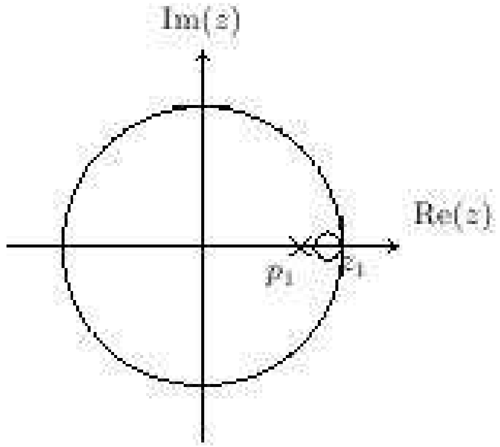

Figure 1: An illustration of the pole p_1 marked with a cross and the zero z_1 marked with a circle for $H(z)$, i.e., the so called pole-zero plot for $H(z)$.

For frequencies ω which are close to zero, $e^{j\omega}$ is much closer to z_1 than p_1, which means that $|H(e^{j\omega})|^2$ is small. For larger frequencies (say $\omega \approx \pi$), $e^{j\omega}$ is only slightly closer to p_1 than z_1 which means that we expect $|H(e^{j\omega})|^2$ to be just a little bit larger than 1. A detailed plot is given in Figure 2 below.

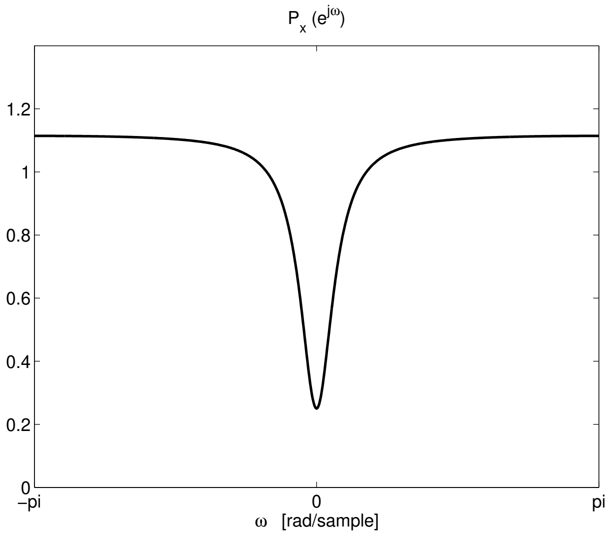

Figure 2: The power spectral density of the ARMA(1,1) process $x[n]$.