MVE136 Random Signals Analysis

Written exam Monday 18 August 2014 2–6 pm

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AIDS: Beta or 2 sheets (4 pages) of hand-written notes (computer print-outs and/or xerox-copies are not allowed), but not both these aids.

GRADES: 12 (40%), 18 (60%) and 24 (80%) points for grade 3, 4 and 5, respectively.

GOOD LUCK!

Task 1. Calculate the probability $\Pr(X(1)=1 \mid X(2)=2)$ for a Poisson process $X(t)$ with arrival rate $\lambda=1$. **(5 points)**

Task 2. Find the autocorrelation $E[X(1)X(2)]$ for a stationary Gaussian random process $X(t)$ such that $\mathrm{E}[X(t)]=0$ and $\mathrm{Var}(X(t))=1$ for all t and $\mathrm{Var}(X(1)+X(2))=3$. **(5 points)**

Task 3. Consider a Markov chain with states $\{0,1,2\}$ that has stationary distribution $\pi = [1/4 \;\; 1/2 \;\; 1/4]$. Find one possible transition matrix P for this Markov chain. **(5 points)**

Task 4. A WSS continuous time bandlimited white noise process $N(t)$ with autocorrelation function $R_{NN}(\tau) = \mathrm{sinc}(\tau/(\pi t_0))$ and PSD $S_{NN}(f) = \pi t_0 \,\mathrm{rect}(f\pi t_0)$ is input to an LTI system with impulse response $h(t) = \mathrm{e}^{-t/t_0}\, u(t)$ and transfer function $H(f) = t_0/(1+j2\pi f t_0)$. Find $\mathrm{E}[Y(t)^2]$ for the output process $Y(t)$ of the LTI system. **(5 points)**

Task 5. Show that the PSD $S_{XX}(f)$ of a continuous time WSS process $X(t)$ is symmetric, that is, show that $S_{XX}(-f) = S_{XX}(f)$. **(5 points)**

Task 6. Compute the periodogram of the data sequence $x[0] = 2$, $x[1] = 0$ and $x[2] = 2$. Try to simplify the expression and plot/sketch it. What is the interpretation of the periodogram?

Obviously, $N = 3$ samples are not enough to estimate the power spectral density accurately. What property of the periodogram would improve as N increases and what weaknesses would remain? **(5 points)**

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Solutions to written exam 18 August 2014

Task 1. $\Pr(X(1)=1|X(2)=2) = \Pr(X(1)=1, X(2)=2)/\Pr(X(2)=2) = \Pr(X(1) = 1, X(2)-X(1)=1)/\Pr(X(2)=2) = \Pr(X(1)=1)\Pr(X(2)-X(1)=1)/\Pr(X(2)=2) = \Pr(X(1)=1)^2/\Pr(X(2)=2) = (\mathrm{e}^{-\lambda\cdot 1}(\lambda\cdot 1)^1/(1!))^2/(\mathrm{e}^{-\lambda\cdot 2}(\lambda\cdot 2)^2/(2!)) = 1/2$.

Task 2. As $X(1)+X(2)$ is zero-mean we have $\mathrm{Var}(X(1)+X(2)) = \mathrm{E}[(X(1)+X(2))^2] = \mathrm{E}[X(1)^2]+\mathrm{E}[X(2)^2]+2\,\mathrm{E}[X(1)X(2)] = \mathrm{Var}(X(1))+\mathrm{Var}(X(2))+2\,\mathrm{E}[X(1)X(2)] = 1+1+2\,\mathrm{E}[X(1)X(2)] = 3$, so that $\mathrm{E}[X(1)X(2)] = 1/2$.

Task 3. It is easy to see that

$$P = \begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 1 & 0 \end{bmatrix}$$

does the job to solve the equation $\pi P = \pi$ for the π given.

Task 4. $R_{YY}(0) = \int_{-\infty}^{\infty} S_{YY}(f)\,df = \int_{-\infty}^{\infty} S_{NN}(f)\,|H(f)|^2\,df = \int_{-\infty}^{\infty} \pi t_0\,\mathrm{rect}\,(f\pi t_0)\,t_0^2/(1+(2\pi f t_0)^2)\,df = \int_{-1/(2\pi t_0)}^{1/(2\pi t_0)} \pi t_0^3/(1+(2\pi f t_0)^2)\,df = (t_0^2/2)\,[\arctan(2\pi f t_0)]_{f=-1/(2\pi t_0)}^{f=1/(2\pi t_0)} = \pi t_0^2/4$.

Task 5. As the autocorrelation function $R_{XX}(\tau)$ is symmetric we have $S_{XX}(-f) = \int_{-\infty}^{\infty} \mathrm{e}^{-j2\pi(-f)\tau} R_{XX}(\tau)\,d\tau = \int_{-\infty}^{\infty} \mathrm{e}^{-j2\pi(-f)(-\hat\tau)} R_{XX}(-\hat\tau)\,d\hat\tau = \int_{-\infty}^{\infty} \mathrm{e}^{-j2\pi f\hat\tau} R_{XX}(\hat\tau)\,d\hat\tau = S_{XX}(f)$.

Task 6. The periodogram is given by $\hat{P}_{per}(e^{j\omega}) = \frac{1}{N}\,|X_N(e^{j\omega})|^2$, where, in this case, $N=3$ and

$$X_N(e^{j\omega}) = X_3(e^{j\omega}) = x[0] + x[1]\,e^{-j\omega} + x[2]\,e^{-2j\omega} = 2\,(1+e^{-2j\omega}).$$

It follows that

$$\hat{P}_{per}(e^{j\omega}) = \frac{4}{3}\,(1+e^{-2j\omega})\,(1+e^{2j\omega}) = \frac{4}{3}\,(2+e^{-2j\omega}+e^{2j\omega}) = \frac{8}{3}\,(1+\cos(2\omega)),$$

which is illustrated in Figure 1 below.

The periodogram is an estimate of the power spectral density and can thus be interpreted as an estimate of the "power per frequency" in the process $x[n]$. We know that the periodogram is asymptotically unbiased and we therefore expect the periodogram

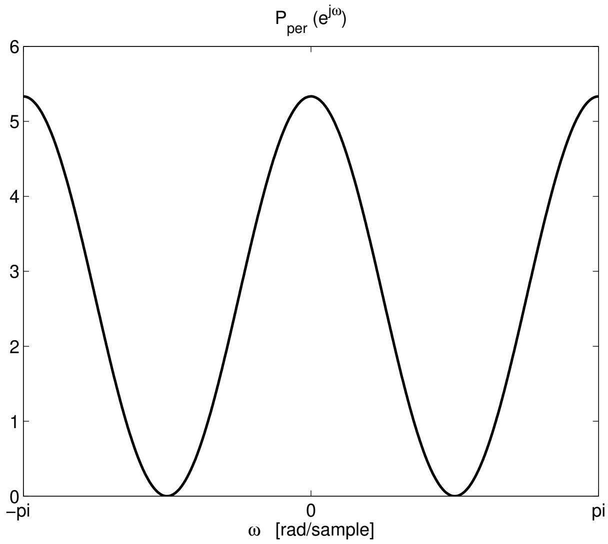

Figure 1: The periodogram of the sequence $x[0], x[1], x[2]$.

to improve as N grows in the sense that the bias goes to zero. However, we also know that the variance converges to a fixed (and often fairly large) value as N grows and the remaining weakness is hence the large variance in the estimates.