MVE136 Random Signals Analysis Fall 2011

Written exam Thursday 20 October 2011 2.00 – 6.00 pm

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AIDS: Beta or 2 sheets (4 pages) of hand-written notes (computer print-outs and/or xerox-copies are not allowed), but not both these aids.

GRADES: 12 (40%), 18 (60%) and 24 (80%) points for grade 3, 4 and 5, respectively.

GOOD LUCK!

Task 1. Let (X,Y) be a continuous random variable with PDF $f_{X,Y}(x,y) = \mathrm{e}^{-x-y-xy} \,/\big(\int_0^\infty (1+z)^{-1}\,\mathrm{e}^{-z}\,dz\big)$ for $x,y \geq 0$ (and 0 otherwise). Find $\mathbf{E}\{X\,|\,Y=y\}$. **(5 points)**

Task 2. Give an example of a WSS random process that is not strict sense stationary. **(5 points)**

Task 3. Find $\mathbf{E}\{X_k\}$ for $k \geq 0$ for a Markov chain $\{X_k\}_{k=0}^{\infty}$ with states $\{0,1,2\}$ and transition probability matrix

$$P = \begin{pmatrix} 1/2 & 1/2 & 0 \\ 0 & 2/3 & 1/3 \\ 1/4 & 0 & 3/4 \end{pmatrix}$$

when the initial distribution $\pi(0)$ is equal to the stationary distribution π. **(5 points)**

Task 4. As you know, the Fourier transform $(\mathcal{F}g)(f)$ and inverse Fourier transform $(\mathcal{F}^{-1}g)(t)$ of a function $g:\mathbb{R}\rightarrow\mathbb{R}$ are given by

$$(\mathcal{F}g)(f) = \int_{-\infty}^{\infty} \mathrm{e}^{-j2\pi ft}\,g(t)\,dt \quad \text{and} \quad (\mathcal{F}^{-1}g)(t) = \int_{-\infty}^{\infty} \mathrm{e}^{j2\pi ft}\,g(f)\,df,$$

respectively, and satisfy $(\mathcal{F}^{-1}\mathcal{F}g)(t) = g(t)$. Prove that $(\mathcal{F}\mathcal{F}g)(t) = g(-t)$. **(5 points)**

Task 5. Show that the output $\{Y(n)\}_{n=-\infty}^{\infty}$ of a discrete time LTI system with impulse response $h(n) = 2\,(-\frac{1}{2})^n - (-\frac{1}{4})^n$ for $n \geq 0$ (and 0 otherwise) is white noise when the input $\{X(n)\}_{n=-\infty}^{\infty}$ is given by $X(n) = e(n) + \frac{3}{4}\,e(n-1) + \frac{1}{8}\,e(n-2)$, where $\{e(n)\}_{n=-\infty}^{\infty}$ is white noise with $\mathbf{E}\{e(n)^2\} = 1$. **(5 points)**

Task 6. We have collected four measurements

$$x[0]=0.3, \quad x[1]=-0.2, \quad x[2]=0.1 \quad \text{and} \quad x[3]=-0.3,$$

of a discrete time WSS random process $\{x[n]\}_{n=-\infty}^{\infty}$ and we wish to find a mathematical model for the measured process. We select to use a simple AR(1)-model $x[n]+a_1\,x[n-1] = e[n]$ for $n\in\mathbb{Z}$, where $\{e[n]\}_{n=-\infty}^{\infty}$ is discrete time white noise: Your task is to estimate the parameter a_1 and the white noise variance $\sigma_e^2 = \mathbf{E}\{e[n]^2\}$. **(5 points)**

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Solutions to written exam Thursday 20 October 2011

Task 1. As $f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)\,dx = \int_0^{\infty} \mathrm{e}^{-x-y-xy}\,dx / \big(\int_0^{\infty}(1+z)^{-1}\,\mathrm{e}^{-z}\,dz\big) = (1+y)^{-1}\,\mathrm{e}^{-y} / \big(\int_0^{\infty}(1+z)^{-1}\,\mathrm{e}^{-z}\,dz\big)$ for $y \geq 0$ (and 0 otherwise) we have $f_{X|Y}(x|y) = f_{X,Y}(x,y)/f_Y(y) = (1+y)\,\mathrm{e}^{-x-xy}$ for $x, y \geq 0$ (and 0 otherwise), so that $\mathbf{E}\{X\,|\,Y=y\} = \int_{-\infty}^{\infty} x\, f_{X|Y}(x|y)\,dx = \int_0^{\infty} x\,(1+y)\,\mathrm{e}^{-x(1+y)}\,dx = \ldots = (1+y)^{-1}$ for $y \geq 0$.

Task 2. For example, the process $\{X(t)\}_{t\in\mathbb{Z}}$ made up of independent random variables that are $\mathrm{N}(0,1)$-distributed for t even and that have a discrete Rademacher distribution with PMF $P_{X(t)}(-1) = P_{X(t)}(1) = 1/2$ for t odd, as this process is zero-mean with autocorrelation function $R_{XX}(\tau) = \delta(\tau)$, but clearly is not strict sense stationary.

Task 3. The stationary distribution $\pi = (\pi_0\ \pi_1\ \pi_2)$ solves the system of equations $\pi = \pi P$ with $\sum_{n=0}^{2} \pi_n = 1$, which readily gives $\pi = (2/9\ \ 1/3\ \ 4/9)$. As $\pi(0) = \pi$ we have $\pi(k) = \pi$ for all $k \geq 0$, so that (with obvious notation) $\mathbf{E}\{X_k\} = \mathbf{E}\{\pi\} = 0\cdot 2/9 + 1\cdot 1/3 + 2\cdot 4/9 = 11/9$ for $k \geq 0$.

Task 4. As $(\mathcal{F}g)(f) = (\mathcal{F}^{-1}g)(-f)$ we have $(\mathcal{F}\mathcal{F}g)(t) = (\mathcal{F}^{-1}\mathcal{F}g)(-t) = g(-t)$.

Task 5. We can view $X(n)$ as the output from an LTI system with input $e(n)$ and impulse response g given by $g(0) = 1$, $g(1) = \frac{3}{4}$, $g(2) = \frac{1}{8}$ and $g(n) = 0$ otherwise. As $S_{ee}(f) = 1$ this gives $S_{XX}(f) = S_{ee}(f)\,|G(f)|^2 = |1 + \frac{3}{4}\,\mathrm{e}^{-j2\pi f} + \frac{1}{8}\,\mathrm{e}^{-j4\pi f}|^2$. Further, we have $S_{YY}(f) = S_{XX}(f)\,|H(f)|^2$, where $H(f) = \sum_{n=-\infty}^{\infty} h(n)\,\mathrm{e}^{-j2\pi nf} = 2\sum_{n=0}^{\infty}(-\frac{1}{2})^n\,\mathrm{e}^{-j2\pi nf} - \sum_{n=0}^{\infty}(-\frac{1}{4})^n\,\mathrm{e}^{-j2\pi nf} = 2/(1+\frac{1}{2}\,\mathrm{e}^{-j2\pi f}) - 1/(1+\frac{1}{4}\,\mathrm{e}^{-j2\pi f}) = 1/\big((1+\frac{1}{2}\,\mathrm{e}^{-j2\pi f})\,(1+\frac{1}{4}\,\mathrm{e}^{-j2\pi f})\big) = 1/(1+\frac{3}{4}\,\mathrm{e}^{-j2\pi f} + \frac{1}{8}\,\mathrm{e}^{-j4\pi f}) = 1/G(f)$.

Task 6. As the autocorrelation function $r_x[k]$ of the AR(1)-process satisfies the Yule-Walker equations

$$a_1\, r_x[0] + r_x[1] = 0 \quad \text{and} \quad r_x[0] + a_1\, r_x[1] = \sigma_e^2,$$

we obtain estimates $\hat{a}_1$ and $\hat{\sigma}_e^2$ of the parameters a_1 and σ_e^2 by solving the equations

$$\hat{a}_1\, \hat{r}_x[0] + \hat{r}_x[1] = 0 \quad \text{and} \quad \hat{r}_x[0] + \hat{a}_1\, \hat{r}_x[1] = \hat{\sigma}_e^2,$$

where

$$\hat{r}_x[0] = \frac{1}{4}\sum_{n=0}^{3} x[n]^2 = \ldots = 0.0575 \quad \text{and} \quad \hat{r}_x[1] = \frac{1}{4}\sum_{n=0}^{2} x[n]\,x[n+1] = \ldots = -0.0275$$

are estimated values of the autocorrelation function. This gives $\hat{a}_1 = 0.0275/0.0575$ and $\hat{\sigma}_e^2 = 0.0575 - 0.0275^2/0.0575$.