MVE136 Random Signals Analysis

Written exam Monday 9 January 2012 8.30 am – 12.30 pm

Teacher and Jour: Patrik Albin, telephone 070 6945709.

Aids: Beta or 2 sheets (4 pages) of hand-written notes (computer print-outs and/or xerox-copies are not allowed), but not both these aids.

Grades: 12 (40%), 18 (60%) and 24 (80%) points for grade 3, 4 and 5, respectively.

Good luck!

Task 1. Calculate $E(X \,|\, X>0)$ for a random variable $X \sim \mathrm{N}(0,1)$. **(5 points)**

Task 2. Let $X(t)$ be a continuous time WSS random process defined for all real times $t \in \mathbb{R}$. Is the time reversed process $Y(t) = X(-t)$ also WSS? (The answer must be motivated!) **(5 points)**

Task 3. In order to find the expected value $E(T)$ of the time $T = \min\{n \in \mathbb{N} : X_n = 2\}$ it takes the discrete time Markov chain $X(n)$ with state space E, initial distribution $\pi(0)$ and transition probability matrix P given by

$$E = \{0,1,2\}, \quad \pi(0) = [1\ \ 0\ \ 0] \quad \text{and} \quad P = \begin{bmatrix} 1/2 & 1/3 & 1/6 \\ 0 & 2/3 & 1/3 \\ 0 & 0 & 1 \end{bmatrix},$$

respectively, to reach the state 2, we notice that $T+1$ has the same distribution as the reccurence time $\hat{T}_2 = \min\{n \geq 1 : \hat{X}(n) = 2\}$ for the Markov chain $\hat{X}(n)$ with state space $\hat{E}$, initial distribution $\hat{\pi}(0)$ and transition probability matrix $\hat{P}$ given by

$$\hat{E} = \{0,1,2\}, \quad \hat{\pi}(0) = [0\ \ 0\ \ 1] \quad \text{and} \quad \hat{P} = \begin{bmatrix} 1/2 & 1/3 & 1/6 \\ 0 & 2/3 & 1/3 \\ 1 & 0 & 0 \end{bmatrix}.$$

Writing $\hat{\pi} = [\hat{\pi}_0\ \ \hat{\pi}_1\ \ \hat{\pi}_2]$ for the stationary distribution of $\hat{X}(n)$, theory says (as well as do heuristics) that $\hat{\pi}_2 = 1/E(\hat{T}_2)$. Use this to calculate $E(T)$. **(5 points)**

Task 4. For which ferquency $f_0 > 1$ does the lowpass WSS random process $X(t)$ with PSD $S_{XX}(f) = 1$ for $|f| \leq f_0$ and $S_{XX}(f) = 0$ otherwise have the same average normalized power $R_{XX}(0)$ as the average normalized power $R_{YY}(0)$ of the bandpass WSS random process $Y(t)$ with PSD $S_{YY}(f) = 1$ for $|f - f_0| \leq 1$, $S_{YY}(f) = 1$ for $|f + f_0| \leq 1$ and $S_{YY}(f) = 0$ otherwise? **(5 points)**

Task 5. Let $e[n]$ be discrete time Gaussian noise with zero mean and unit variance. Given a constant $a \in (-1,1)$, how can the Fourier transform (/frequency analysis) techniques of Chapter 11 in the book be employed to establish that the discrete time random process $X[n] = \sum_{k=0}^{\infty} a^k\, e[n-k]$ has autocorrelation function $R_{XX}[n] = a^{|n|}/(1-a^2)$? (The required calculations need not be carried out in full detail - it is sufficient to just outline what should be done.) **(5 points)**

Task 6. Explain the ideas behind Blackman-Tukey's method either in the time domain or in the frequency domain. Also, what trade-off do you have to make when you set the width of the window (often denoted M) in the time domain? **(5 points)**

MVE136 Random Signals Analysis

Solutions to written exam Monday 9 January 2012

Task 1. We have $f_{X|X>0}(x) = f_X(x)/P(X>0) = \frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-x^2/2}/(1/2) = \sqrt{2/\pi}\,\mathrm{e}^{-x^2/2}$ for $x>0$, so that $E(X\,|\,X>0) = \int_{-\infty}^{\infty} x\, f_{X|X>0}(x)\,dx = \int_0^{\infty} x\,\sqrt{2/\pi}\,\mathrm{e}^{-x^2/2}\,dx = \sqrt{2/\pi}$.

Task 2. We have $\mu_Y(t) = E(Y(t)) = E(X(-t)) = \mu_X(-t) = \mu_X =$ constant and $R_{YY}(t,t{+}\tau) = E(Y(t)Y(t{+}\tau)) = E(X(-t)X(-(t{+}\tau))) = R_{XX}(-t,-(t{+}\tau)) = R_X(-(t+\tau)-(-t)) = R_X(-\tau) = R_X(\tau)$ a function of τ only, so that $Y(t)$ is also WSS.

Task 3. We find $\hat{\pi}$ as the PMF on $\hat{E}$ that solves the equation $\hat{\pi}\hat{P} = \hat{\pi}$. As this gives $\hat{\pi} = [\frac{2}{5}\ \frac{2}{5}\ \frac{1}{5}]$ it follows that $\mathbf{E}\{T\} = \mathbf{E}\{\hat{T}_2\} - 1 = 1/\hat{\pi}_2 - 1 = 5 - 1 = 4$.

Task 4. As $R_{XX}(0) = \int_{\infty}^{\infty} S_{XX}(f)\,df = 2\,f_0$ and $R_{YY}(0) = \int_{\infty}^{\infty} S_{YY}(f)\,df = 4$ we must have $f_0 = 2$.

Task 5. See Example 11.3 in the book.

Task 6. In the time-domain BT's algorithm can be motivated as follows: Estimates of the autocorrelation function $\hat{r}_x[k] = \frac{1}{N}\sum_{n=k}^{N-1} x[n]\,x[n-k]$ are less reliable for large time lags k as such lags have smaller sample support. To reduce the variance we therefore give them a smaller weight when we compute the periodogram $\hat{P}_{\mathrm{BT}}(\mathrm{e}^{j\omega}) = \sum_{k=-M}^{M} w_{\mathrm{lag}}[k]\,\hat{r}[k]\,\mathrm{e}^{-jk\omega}$, where $w_{\mathrm{lag}}[k]$ is the weighting window.

In the frequency domain BT's algorithm can be motivated as follows: As $X_N(\mathrm{e}^{j\omega_1})$ and $X_N(\mathrm{e}^{j\omega_2})$ are approximately uncorrelated when $|\omega_1{-}\omega_2|/N$ is small, it is reasonable to assume that an averaging in the frequence domain $\hat{P}_{\mathrm{BT}}(\mathrm{e}^{j\omega}) = \frac{1}{2\pi}\,\hat{P}_{\mathrm{per}}(\mathrm{e}^{j\omega}){\star}W_{\mathrm{lag}}(\mathrm{e}^{j\omega})$ can be used to reduce the variance. From this intuitive argumentation we also understand that the width of $W_{\mathrm{lag}}(\mathrm{e}^{j\omega})$ can be reduced if N is increased (larger N yields less correlation between adjacent frequencies).

A narrow window (small M) in the time domain gives a small variance at the cost of a larger bias. Naturally, a wide window instead gives larger variance but smaller bias. We call this a bias-variance trade-off. Of course, a narrow window in the time domain corresponds to a wide window in the frequency domain, in case you prefer to discuss the trade-off in the frequency domain instead.