MVE136 Random Signals Analysis

Written exam Thursday 16 January 2014 2 – 6 pm

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AIDS: Beta or 2 sheets (4 pages) of hand-written notes (computer print-outs and/or xerox-copies are not allowed), but not both these aids.

GRADES: 12 (40%), 18 (60%) and 24 (80%) points for grade 3, 4 and 5, respectively.

GOOD LUCK!

Task 1. Consider the discrete-time random process $\{X_n,\, n \geq 1\}$ given by $X_n = \sin(nU)$ for $n \geq 1$, where U is a random variable that is uniformly distributed over the interval $[-\pi, \pi]$. Show that the process X_n is wide-sense stationary (WSS). [Hint: The formula $\sin(x)\sin(y) = \frac{1}{2}\cos(x-y) - \frac{1}{2}\cos(x+y)$ can be useful.] **(5 points)**

Task 2. The one and same continuoius-time zero-mean wide-sense stationary (WSS) random process $\{X(t),\, t \in \mathbb{R}\}$ with autocorrelation function $R_{XX}(s,t) = \mathrm{e}^{-|t-s|}$ is (at the same time) input to two different continuous-time linear time-invariant (LTI) systems with outputs $\{Y_1(t),\, t \in \mathbb{R}\}$ and $\{Y_2(t),\, t \in \mathbb{R}\}$, respectively, and impulse responses $h_1(t) = \mathrm{e}^{-2t}$ for $t \geq 0$, $h_1(t) = 0$ for $t < 0$ and $h_2(t) = \mathrm{e}^{-3t}$ for $t \geq 0$, $h_2(t) = 0$ for $t < 0$, respectively. Find $E(Y_1(t)Y_2(t))$. **(5 points)**

Task 3. Find the probability density function (PDF) for the random variable $X(0) + X(t)$ when $\{X(t),\, t \in \mathbb{R}\}$ is a stationary continuous-time zero-mean Gaussian random process with power spectral density (PSD) $S_{XX}(f) = 1/(1+(\pi f)^2)$. **(5 points)**

Task 4. Calculate $E(X[n]X[n+1])$ for non-negative integers n when $\{X[n],\, n \geq 0\}$ is a Markov chain with state space (/possible values) S, initial distribution $\pi(0)$ and transition probability matrix P, respectively, given by

$$S = \{0, 1\}, \qquad \pi(0) = [1/2 \;\; 1/2] \qquad \text{and} \qquad P = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}. \qquad \textbf{(5 points)}$$

Task 5. Consider a Poisson process $\{X(t),\, t \geq 0\}$ with arrival rate (/intensity) $\lambda > 0$. Is it possible to find two non-random functions $f(t)$ and $g(t)$ such that the process $\{Y(t),\, t > 0\}$ given by $Y(t) = f(t)X(t) - g(t)$ is a zero-mean wide-sense stationary (WSS) process? [Hint: Recall that $E(X(t)) = Var(X(t)) = \lambda t$ and $C_{XX}(s,t) = \lambda \min(s,t)$.]

(5 points)

Task 6. *Background information:* The Wiener filter is not only useful for noise suppression but can also be used to do, e.g., de-blurring of images or echo cancellation of sounds. In this problem we study an example of that type.

Problem statement: Suppose that we observe the output $x[n]$ from $H_0(z) = 1{+}0.7\,z^{-1}$ when the signal of interest $d[n]$ is the filter input, see Figure 1 below.

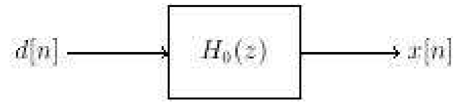

Figure 1: An illustration of the relation between $d[n]$ and $x[n]$.

The input signal is a wide sense stationary (WSS) signal with power spectral density $P_d(e^{j\omega})$ and we note from Figure 1 that there is no noise. Find the frequency response $H(e^{j\omega})$ of the non-causal IIR Wiener filter (sometimes also known as the Wiener smoother). As always, the output $\hat{d}[n]$ of the Wiener filter minimizes mean square error (MSE), $E\{(\hat{d}[n] - d[n])^2\}$, see Figure 2 below.

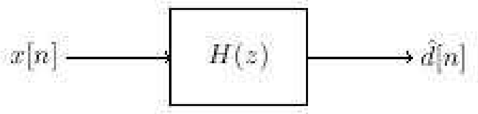

Figure 2: An illustration of the input and output of the Wiener filter. We design the filter in order to obtain $\hat{d}[n] \approx d[n]$.

What is the MSE of the filter that you derived above?

Hint: It follows from how $x[n]$ is generated that $P_{dx}(e^{j\omega}) = \overline{H_0(e^{j\omega})}\,P_d(e^{j\omega})$ and $P_x(e^{j\omega}) = |H_0(e^{j\omega})|^2 P_d(e^{j\omega})$, where the bar on top of H_0 denotes complex conjugate.

(5 points)

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Solutions to written exam Thursday 16 January 2014

Task 1. Clearly $E(X_n) = 0$ and $E(X_n X_{n+m}) = \frac{1}{2}E(\cos(mU)) - \frac{1}{2}E(\cos((2n+m)U)) = \frac{1}{2}\delta(m) + 0$ do not depend on n so that X_n is WSS.

Task 2. We have $E(Y_1(t)Y_2(t)) = E\big(\int_{-\infty}^{\infty} h_1(u)X(t-u)\,du \int_{-\infty}^{\infty} h_2(v)X(t-v)\,dv\big) = E\big(\int_0^{\infty} \mathrm{e}^{-2u}X(t-u)\,du \int_0^{\infty} \mathrm{e}^{-3v}X(t-v)\,dv\big) = \int_0^{\infty}\int_0^{\infty} \mathrm{e}^{-2u}\mathrm{e}^{-3v}E(X(t-u)X(t-v))\,dudv = \int_0^{\infty}\int_0^{\infty} \mathrm{e}^{-2u}\mathrm{e}^{-3v}\mathrm{e}^{-|u-v|}\,dudv = \int_{u=0}^{u=\infty}\int_{v=u}^{v=\infty} \mathrm{e}^{-2u}\mathrm{e}^{-3v}\mathrm{e}^{-(v-u)}\,dvdu + \int_{v=0}^{v=\infty}\int_{u=v}^{u=\infty} \mathrm{e}^{-2u}\mathrm{e}^{-3v}\mathrm{e}^{-(u-v)}\,dudv = \int_{u=0}^{u=\infty} \mathrm{e}^{-u}\int_{v=u}^{v=\infty} \mathrm{e}^{-4v}\,dvdu + \int_{v=0}^{v=\infty} \mathrm{e}^{-2v}\int_{u=v}^{u=\infty} \mathrm{e}^{-3u}\,dudv = \int_{u=0}^{u=\infty} \frac{1}{4}\mathrm{e}^{-5u}\,du + \int_{v=0}^{v=\infty} \frac{1}{3}\mathrm{e}^{-5v}\,dv = \frac{1}{20} + \frac{1}{15} = \frac{7}{60}$.

Task 3. The given PSD corresponds to the autocorrelation function $R_{XX}(t) = \mathrm{e}^{-2|t|}$. Therefore $X(0)+X(t)$ is a zero-mean Gaussian (/normal distributed) random variable with variance $Var(X(0)+X(t)) = E((X(0)+X(t))^2) = E(X(0)^2) + 2\,E(X(0)X(t)) + E(X(t)^2) = R_{XX}(0) + 2\,R_{XX}(t) + R_{XX}(0) = 2\,(1+\mathrm{e}^{-2|t|})$.

Task 4. As the only possible values of the random variable $X[n]X[n+1]$ are 0 and 1, we have $E(X[n]X[n+1]) = 0\cdot P(X[n]X[n+1] = 0) + 1\cdot P(X[n]X[n+1] = 1) = P(X[n]X[n+1] = 1) = P(X[n] = 1,\, X[n+1] = 1) = \pi(n)_1 P_{11} = \pi(n)_1(1/2)$. Noting that $\pi(0) = \pi$ is in fact a stationary distribution for the chain we have $\pi(n) = \pi(0) = \pi$, so that $\pi(n)_1 = \pi(0)_1 = 1/2$ and $E(X[n]X[n+1]) = 1/4$. (The latter result is in fact also more or less obviously true already from the beginning without calculations from symmetry considerations ... .)

Task 5. In order for $Y(t)$ to be WSS $Var(Y(t))$ must be constant (not depending on t). As it is also requested that $Y(t)$ is zero-mean it follows that $E(Y(t)) = f(t)\,\lambda t - g(t) = 0$ and $Var(Y(t)) = f(t)^2\lambda t = C$, where $C \geq 0$ is a constant. Hence we must have $f(t) = \pm C/\sqrt{\lambda t}$ and $g(t) = \pm C\sqrt{\lambda t}$. From this it follows that $C_{YY}(s,t) = f(s)f(t)\,C_{XX}(s,t) = \min(s,t)/\sqrt{st} = \min(\sqrt{s/t}, \sqrt{t/s})$, which is not a function of $t-s$ only. Therefore it is not possible to find functions $f(t)$ and $g(t)$ such that $Y(t) = f(t)X(t) - g(t)$ is a zero-mean WSS process.

Task 6. We know from the lecture notes that the non-causal IIR Wiener filter is given by $H(e^{j\omega}) = P_{dx}(e^{j\omega})/P_x(e^{j\omega})$, and by plugging in the expressions given in the problem formulation we get

$$H(e^{j\omega}) = \frac{\overline{H_0(e^{j\omega})}\, P_d(e^{j\omega})}{|H_0(e^{j\omega})|^2 P_d(e^{j\omega})} = \frac{1}{H_0(e^{j\omega})} = \frac{1}{1+0.7\,e^{-j\omega}}.$$

The final solution is thus simply that $H(e^{j\omega}) = 1/H_0(e^{j\omega})$, which means that the frequency response from $d[n]$ to $\hat{d}[n]$ is $H_0(e^{j\omega})/H_0(e^{j\omega}) = 1$, see Figure 3 below.

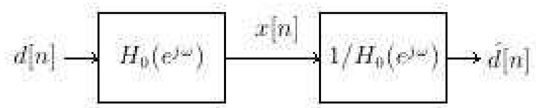

Figure 3: An illustration of the relations between $d[n]$, $x[n]$ and $\hat{d}[n]$.

Clearly, this proves that $\hat{d}[n] = d[n]$ and that the MSE of the filter is zero.

For a simple example like this one, it can be understood that the optimal filter is $H(e^{j\omega}) = 1/H_0(e^{j\omega})$ without making use of the Wiener-Hopf equations. However, it good to know that the Wiener filter can also this type of signal distortion. It is also straightforward to handle combinations of signal distortions and noise disturbances using a Wiener filter, which is a problem that has a much more complicated solution.

A word of caution: It is possible to design examples where the inverse filter $H(z) = 1/H_0(z)$ has a region of convergence (ROC) that does not include the unit circle, which would mean that the Fourier transform does not exist. In this case, however, the only pole of $H(z)$ is at $z = -0.7$ and the ROC$= \{z : |z| > 0.7\}$ which does include the unit circle.