

## MVE136 Random Signals Analysis

Written exam Monday 5 January 2015 2 – 6 pm

TEACHER AND JOUR: Patrik Albin, telephone 070 6945709.

AIDS: Beta or 2 sheets (4 pages) of hand-written notes (computer print-outs and/or xerox-copies are not allowed), but not both these aids.

GRADES: 12 (40%), 18 (60%) and 24 (80%) points for grade 3, 4 and 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

**Task 1.** Consider the discrete time random process given by  $X[0] = 0$  and  $X[n] = \sum_{k=1}^n W_k$  for  $n = 1, 2, \dots$ , where  $W_1, W_2, W_3, \dots$  are independent identically distributed zero-mean unit-variance random variables. Find the ACF  $R_{XX}(m, n)$ . **(5 points)**

**Task 2.** Consider the continuous time random process  $X(t) = A \cos(\omega_0 t) + B \sin(\omega_0 t)$  for  $t \in \mathbb{R}$ , where  $A$  and  $B$  are independent zero-mean unit-variance Gaussian random variables and  $\omega_0 \in \mathbb{R}$  is a constant. Find the joint PDF  $f_{X(s), X(t)}(x, y)$  for the process values  $(X(s), X(t))$ . [Hint:  $\cos(x) \cos(y) + \sin(x) \sin(y) = \cos(x - y)$ .] **(5 points)**

**Task 3.** Prove that  $\pi(k) = \pi(0) P^k$  for a Markov chain  $X[k]$  with transition probability matrix  $P$  and distribution at time  $k$  given by the row matrix  $\pi(k)$ . **(5 points)**

**Task 4.** Consider the continuous time random process  $X(t) = A \sin(\omega_0 t + \Theta)$  for  $t \in \mathbb{R}$ , where  $A$  and  $\Theta$  are independent random variables with  $\Theta$  uniformly distributed over the interval  $[0, 2\pi]$  and  $\omega_0 \in \mathbb{R}$  is a constant. Find the PSD  $S_{XX}(f)$  of  $X(t)$ . [Hint:  $\sin(x) \sin(y) = \frac{1}{2} \cos(x - y) - \frac{1}{2} \cos(x + y)$ .] **(5 points)**

**Task 5.** A zero-mean WSS continuous time process  $Z(t)$  with PSD  $S_{ZZ}(f)$  is sent on a noisy channel where the noise  $N(t)$  is independent of  $Z(t)$  and zero-mean WSS with PSD  $S_{NN}(f)$ . The received signal  $X(t) = Z(t) + N(t)$  is input to a Wiener filter with output  $Y(t)$ . Express  $E[(Z(t) - Y(t))^2]$  in terms of  $S_{ZZ}(f)$  and  $S_{NN}(f)$ . **(5 points)**

**Task 6.** Consider FIR Wiener filtering when the desired signal is generated as  $d[n] = e[n] - 0.5e[n-1]$ , where  $e[n]$  is a zero-mean white noise (wide sense stationary) process with variance  $\sigma_e^2 = 1$ . We observe  $x[n] = d[n] + v[n]$ , where  $v[n]$  is zero-mean white noise with variance  $\sigma_v^2 = 0.5$  which is uncorrelated with  $e[n]$ . Your task is to find the optimal filtering coefficients  $h_0$  and  $h_1$  which are such that  $\hat{d}[n] = h_0 x[n] + h_1 x[n-1]$  minimizes the mean-square distance  $E[(d[n] - \hat{d}[n])^2]$ . **(5 points)**

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## Solutions to written exam 5 January 2015

**Task 1.** According to Example 8.14 in the book by Miller and Childers we have  $R_{XX}(m, n) = \min(m, n)$ .

**Task 2.** This is the special case of Example 8.23 in the book by Miller and Childers with  $\sigma^2 = 1$ .

**Task 3.** This is Equation 9.12 in the book by Miller and Childers.

**Task 4.** According to Example 10.2 in the book by Miller and Childers we have  $S_{XX}(f) = \frac{1}{4} E[A^2] (\delta(f - f_0) + \delta(f + f_0))$ , where  $f_0 = \omega_0/(2\pi)$ .

**Task 5.** By basic calculations we have  $E[(Z(t) - Y(t))^2] = \int_{-\infty}^{\infty} [|H(f)|^2 (S_{ZZ}(f) + S_{NN}(f)) - 2H(f)S_{ZZ}(f) + S_{ZZ}(f)] df$ . Upon insertion of the Wiener filter transfer function  $H(f) = S_{ZZ}(f)/(S_{ZZ}(f) + S_{NN}(f))$  basic algebraic manipulations give  $E[(Z(t) - Y(t))^2] = \int_{-\infty}^{\infty} S_{ZZ}(f)S_{NN}(f)/(S_{ZZ}(f) + S_{NN}(f)) df$ .

**Task 6.** Let us first rederive the Wiener-Hopf equations. According to the orthogonality principle, the optimal filter should satisfy

$$E[(d[n] - \hat{d}[n])x[n]] = 0 \quad \text{and} \quad E[(d[n] - \hat{d}[n])x[n-1]] = 0.$$

By incorporating the expression  $\hat{d}[n] = h_0 x[n] + h_1 x[n-1]$  we obtain

$$h_0 r_x[0] + h_1 r_x[1] = r_{dx}[0] \quad \text{and} \quad h_0 r_x[1] + h_1 r_x[0] = r_{dx}[1].$$

To find the optimal filtering coefficients we therefore need to find  $r_{dx}[0]$ ,  $r_{dx}[1]$ ,  $r_x[0]$  and  $r_x[1]$ : Considering that  $v[n]$  and  $e[n]$  are uncorrelated processes it holds that  $v[n]$  and  $d[n]$  are uncorrelated, so that

$$\begin{aligned} r_{dx}[k] &= E[d[n]x[n-k]] = E[d[n](d[n-k] + v[n-k])] = r_d[k] \\ r_x[k] &= E[(d[n] + v[n])(d[n-k] + v[n-k])] = r_d[k] + r_v[k] \end{aligned}$$

for  $k = 0, 1$ . As we know that  $r_v[0] = \sigma_v^2 = 0.5$  and  $r_v[1] = 0$  it only remains to compute  $r_d[0]$  and  $r_d[1]$ . We can compute these directly from the fact that

$$r_d[k] = E[d[n]d[n-k]] = E[(e[n] + 0.5e[n-1])(e[n-k] + 0.5e[n-k-1])]$$

for  $k = 0, 1$ , which yields  $r_d[0] = 1.25\sigma_e^2 = 1.25$  and  $r_d[1] = -0.5\sigma_e^2 = 0.5$ . Hence we have

$$h_0 1.75 + h_1 (-0.5) = 1.25 \quad \text{and} \quad h_0 (-0.5) + h_1 1.75 = (-0.5),$$

from which we can find the solutions  $h_0 = \frac{31}{45} \approx 0.689$  and  $h_1 = -\frac{4}{45} \approx -0.089$ .