MVE136 Random Signals Analysis

Written exam Monday 17 August 2015 2 − 6 pm

TEACHER AND JOUR: Patrik Albin, telephone 070 6945709.

AIDS: Beta or 2 sheets (4 pages) of hand-written notes (computer print-outs and/or xerox-copies are not allowed), but not both these aids.

GRADES: 12 (40%), 18 (60%) and 24 (80%) points for grade 3, 4 and 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Task 1. Find $\Pr(X(1)=1)$ for a Poisson process $X(t)$ with $E[X(2)^2]=6$. **(5 points)**

Task 2. Let $X(t)$ be a WSS Gaussian process such that $\Pr\big(X(1)+X(2)\leq 3\big)=\Phi(1/3)$ and $\Pr\big(X(1)+X(2)\leq 4\big)=\Phi(1/2)$, where as usual $\Phi(x)$ denotes the standard Gaussian CDF. Find the mean function $\mu_X(t)=E[X(t)]$ of the process $X(t)$. **(5 points)**

Task 3. Consider a Markov chain X_k with states 0 and 1, and with $X_0=0$. Find the transition matrix P if it is known that $E[X_1]=E[X_2]=1/2$. **(5 points)**

Task 4. Write a little essay providing the basic ideas and techniques of spectral estimation for WSS random processes. **(5 points)**

Task 5. A continuous time LTI system has input $X(t)=U\cos(\omega_0 t)+V\sin(\omega_0 t)$, where U and V are independent standard normal random variables and ω_0 is a constant. Find the impulse response $h(t)$ if the output is $Y(t)=U\sin(\omega_0 t)-V\cos(\omega_0 t)$. **(5 points)**

Task 6. Your task here is to derive the Wiener-Hopf equations for an FIR-filter of order 2. To be more specific, suppose that the impulse response of a linear filter is

$$h[n]=h_0\,\delta[n]+h_1\,\delta[n-1]+h_2\,\delta[n-2],$$

and that the input signal is a WSS signal $x[n]$. Derive a matrix equation that determines the filter coefficients h_0, h_1 and h_2 so that the output from the filter $\hat{d}[n]$ minimizes $\mathrm{E}\left[(d[n]-\hat{d}[n])^2\right]$. See Figure 1 below for an illustration. You may use the orthogonality principle to derive the equation asked for if you want. **(5 points)**

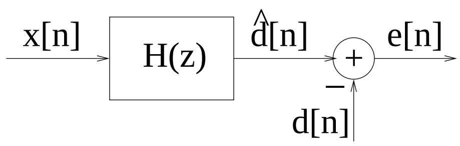

Figure 1: An illustration of the relation between $x[n]$, $h[n]$, $\hat{d}[n]$, $d[n]$ and the filtering error $e[n]$ that we seek to minimize.

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Solutions to written exam 17 August 2015

Task 1. As $E[X(2)^2] = 2\lambda + (2\lambda)^2 = (2\lambda + 1/2)^2 - 1/4$ equals $6 = 24/4$ we have $2\lambda = -1/2 \pm \sqrt{1/4+24/4} = -1/2 \pm 5/2 = 2$, so that $\lambda = 1$ and $\Pr(X(1) = 1) = \lambda\,\mathrm{e}^{-\lambda} = 1/\mathrm{e}$.

Task 2. As $X(t)$ is WSS the mean function is constant $\mu_X(t) = \mu_X$. As $X(1)+X(2)$ is $N(m, \sigma^2)$-distributed with $m = E[X(1)+X(2)] = 2\mu_X$, we have $\Pr\big(X(1)+X(2) \le 3\big) = \Phi\big((3-2\mu_X)/\sigma\big)$ and $\Pr\big(X(1)+X(2) \le 4\big) = \Phi\big((4-2\mu_X)/\sigma\big)$, so that $(3-2\mu_X)/\sigma = 1/3$ and $(4-2\mu_X)/\sigma = 1/2$, giving $\mu_X = 1/2$ (and $\sigma = 6$).

Task 3. As $E[X_1] = 0 \cdot p_{0,0} + 1 \cdot p_{0,1} = p_{0,1} = 1/2$ we have $p_{0,0} = 1 - p_{0,1} = 1/2$. As $E[X_2] = 0 \cdot \Pr(X_2 = 0) + 1 \cdot \Pr(X_2 = 1) = \Pr(X_2 = 1) = ([1\ \ 0]\, P^2)_1 = \ldots = 1/4 + p_{1,1}/2 = 1/2$ we have $p_{1,1} = 1/2$ and $p_{1,0} = 1 - p_{1,1} = 1/2$.

Task 4. See Section 10.4 in the book by Miller and Childers.

Task 5. It is easy to see mathematically or figure out more "hands-on" that the impulse response must be $h(t) = \delta(t - \pi/(2\omega_0))$.

Task 6. According to the orthogonality principle, the optimal filter must satisfy

$$\mathrm{E}\left[(d[n]-\hat{d}[n])\,x[n]\right] = \mathrm{E}\left[(d[n]-\hat{d}[n])\,x[n-1]\right] = \mathrm{E}\left[(d[n]-\hat{d}[n])\,x[n-2]\right] = 0,$$

since the error $d[n] - \hat{d}[n]$ should be orthogonal to the data $x[n]$, $x[n-1]$ and $x[n-2]$. By substituting $\hat{d}[n] = h_0\, x[n] + h_1\, x[n-1] + h_2\, x[n-2]$ above we readily obtain

$$\begin{bmatrix} r_x[0] & r_x[1] & r_x[2] \\ r_x[1] & r_x[0] & r_x[1] \\ r_x[2] & r_x[1] & r_x[0] \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} r_{dx}[0] \\ r_{dx}[1] \\ r_{dx}[2] \end{bmatrix}.$$

These are the Wiener-Hopf equations that we were asked to derive. Instead of using the orthogonality principle, it is also possible to derive the Wiener-Hopf equations by setting the derivatives of our cost function to zero,

$$\frac{\partial}{\partial h_i}\,\mathrm{E}\left[(d[n] - \hat{d}[n])^2\right] = 0 \quad \text{for } i = 0, 1, 2.$$