Solutions to Chapter 4, 5 and 6 Exercises

Problem 4.13

We know the pdf of a distribution can be written as sum of the conditional pdfs.

$$\begin{aligned}
f_X(x) &= \sum_{i=1}^{n} f_{X|A_i}(x) Pr(A_i) \\
E[X] &= \int_{-\infty}^{\infty} x f_X(x)\, dx \\
&= \int_{-\infty}^{\infty} x \sum_{i=1}^{n} f_{X|A_i}(x) Pr(A_i)\, dx
\end{aligned}$$

We can interchange the operations of the integration and summation as they are linear and rewrite the above equation as

$$\begin{aligned}
E[X] &= \sum_{i=1}^{n} \left(\int_{-\infty}^{\infty} x f_{X|A_i}(x) Pr(A_i)\, dx \right) \\
E[X] &= \sum_{i=1}^{n} \left(Pr(A_i) \int_{-\infty}^{\infty} x f_{X|A_i}(x)\, dx \right) \\
E[X] &= \sum_{i=1}^{n} Pr(A_i) E[X|A_i]
\end{aligned}$$

Problem 4.15

$$f_\Theta(\theta) = \frac{1}{2\pi}$$

(a) $Y = \sin\theta$. This equation has two roots. At θ and $\pi - \theta$

$$\begin{aligned}
f_Y(y) &= \sum_{\theta_i} f_\Theta(\theta) \left|\frac{d\theta}{dy}\right|_{\theta=\theta_i} \\
&= \left|\frac{1}{2\pi\cos\theta}\right|_{\theta=\theta} + \left|\frac{1}{2\pi\cos\theta}\right|_{\theta=\pi-\theta} \\
&= \frac{1}{\pi\cos\theta} \\
&= \frac{1}{\pi\sqrt{1-y^2}}
\end{aligned}$$

(b) $Z = \cos\theta$. This equation has two roots. At θ and $2\pi - \theta$

$$\begin{aligned}
f_Z(z) &= \sum_{\theta_i} f_\Theta(\theta) \left|\frac{d\theta}{dy}\right|_{\theta=\theta_i} \\
&= \left|\frac{1}{2\pi\sin\theta}\right|_{\theta=\theta} + \left|\frac{1}{2\pi\sin\theta}\right|_{\theta=2\pi-\theta} \\
&= \frac{1}{\pi\sin\theta} \\
&= \frac{1}{\pi\sqrt{1-y^2}}
\end{aligned}$$

(c) $W = \tan\theta$. This equation has two roots. At θ and $\pi + \theta$

$$\begin{aligned}
f_W(w) &= \sum_{\theta_i} f_\Theta(\theta) \left|\frac{d\theta}{dy}\right|_{\theta=\theta_i} \\
&= \left|\frac{1}{2\pi\sec^2\theta}\right|_{\theta=\theta} + \left|\frac{1}{2\pi\sec^2\theta}\right|_{\theta=\pi+\theta} \\
&= \frac{1}{\pi\sec^2\theta} \\
&= \frac{1}{\pi(\tan^2\theta+1)} \\
&= \frac{1}{\pi(w^2+1)}
\end{aligned}$$

Problem 4.20

$$
\begin{cases}
P_Y(y=-2) = P_X(x<-1) = \mathrm{Q}\left(\dfrac{1}{\sigma_X}\right) & (1)\\
P_Y(-2 \le y \le 2) = P_X(-1 \le x \le 1) & (2)\\
P_Y(y=2) = P_X(x>1) = \mathrm{Q}\left(\dfrac{1}{\sigma_X}\right) & (3)
\end{cases}
$$

For middle part (2), Y=2X, so we can write:

$$
\begin{aligned}
f_Y(y) &= f_X(x)\left|\frac{dx}{dy}\right|\Bigg|_{x=\frac{1}{2}y}\\
&= \frac{1}{2} f_X\left(\frac{1}{2}y\right)\\
&= \frac{1}{2\sqrt{2\pi\sigma_X^2}} \exp\left(\frac{-\left(\frac{1}{2}y\right)^2}{2\sigma_X^2}\right)\\
&= \frac{1}{2\sqrt{2\pi\sigma_X^2}} \exp\left(\frac{-y^2}{8\sigma_X^2}\right)
\end{aligned}
$$

Finally we get:

$$
\begin{aligned}
&Y \in [-2,2]\\
&f_Y(y) = Q(\frac{1}{\sigma_X})\delta(y+2) + \frac{1}{2\sqrt{2\pi\sigma_X^2}} \exp\left(\frac{-y^2}{8\sigma_X^2}\right) + Q(\frac{1}{\sigma_X})\delta(y-2)\\
&\quad = Q(\frac{1}{\sigma_X})[\delta(y+2)+\delta(y-2)] + \frac{1}{2\sqrt{2\pi\sigma_X^2}} \exp\left(\frac{-y^2}{8\sigma_X^2}\right)
\end{aligned}
$$

Attention please!

For Pr(Y=±2), we have two separate parts. One part is included in (1) and (3) and second part in (2), therefore the step functions which we talked about them in exercise session, are not necessary as far as we know Y is in [-2,2].

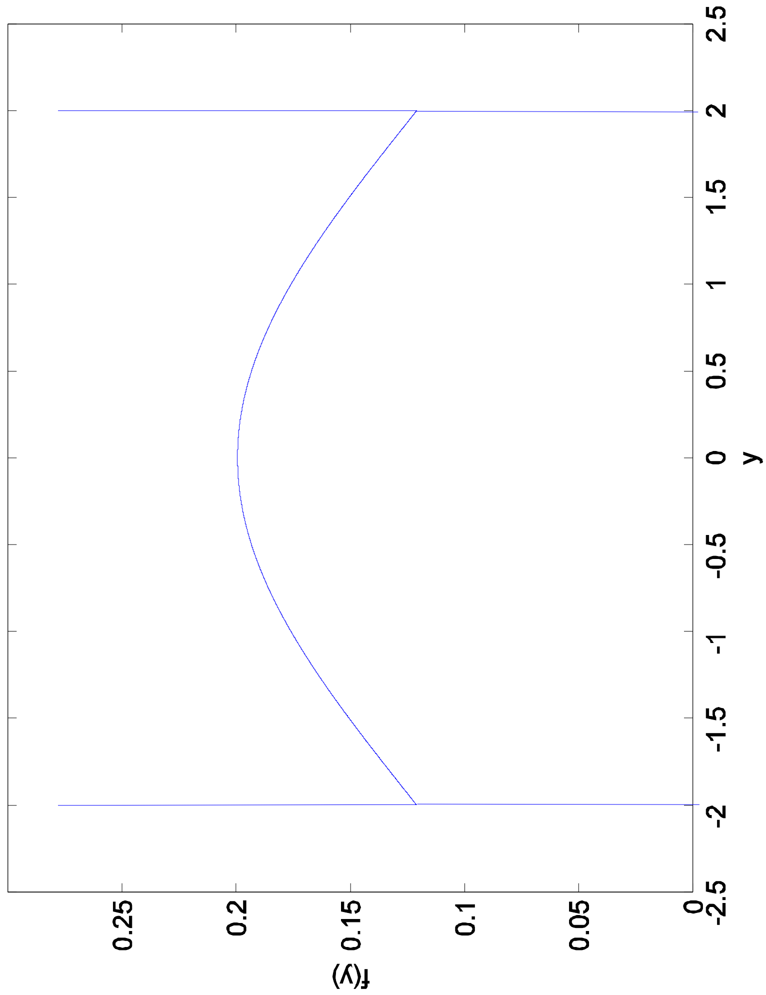

Figure 1

Problem 4.22

$a) X \in [-0.5, 0.5)$

$b)$ X is unform over $[-0.5, 0.5)$

$$f_X(x) = \frac{1}{0.5-(-0.5)} = 1$$

$$c) E\left[X^2\right] = \int_{-\infty}^{+\infty} x^2 f_X(x)\, dx = \int_{-0.5}^{+0.5} x^2 dx = \frac{x^3}{3}\Big|_{-0.5}^{0.5} = \frac{1}{12}$$

Problem 4.31

(a) Characteristic Function

$$\begin{aligned}
f_X(x) &= \frac{1}{2b} exp(-\frac{|x|}{b}) \\
\Phi_X(\omega) &= \int_{-\infty}^{\infty} \frac{1}{2b} e^{-\frac{|x|}{b}} e^{j\omega x} \\
&= \int_{-\infty}^{\infty} \frac{1}{2b} e^{-\frac{|x|}{b}} e^{j\omega x} \\
&= \int_{-\infty}^{0} \frac{1}{2b} e^{\frac{x}{b}} e^{j\omega x} + \int_{0}^{\infty} \frac{1}{2b} e^{-\frac{x}{b}} e^{j\omega x} \\
&= \frac{1}{2b} \left(\left[\frac{e^{\frac{x}{b}+j\omega x}}{\frac{1}{b}+j\omega} \right] \Bigg|_{x=-\infty}^{x=0} + \left[\frac{e^{-\frac{x}{b}+j\omega x}}{-\frac{1}{b}+j\omega} \right] \Bigg|_{x=0}^{x=\infty} \right) \\
&= \frac{1}{2b} \left(\frac{1}{\frac{1}{b}+j\omega} - \frac{1}{-\frac{1}{b}+j\omega} \right) \\
&= \frac{1}{2b} \left(\frac{1}{\frac{1}{b}+j\omega} + \frac{1}{\frac{1}{b}-j\omega} \right) \\
&= \frac{1}{1+b^2\omega^2}
\end{aligned}$$

(b) Taylor Series Expansion of $\Phi_X(\omega)$.

$$\begin{aligned}
\Phi_X(\omega) &= \frac{1}{1+b^2\omega^2} \\
&= \sum_{k=0}^{\infty} (-1)^k (b\omega)^{2k}
\end{aligned}$$

(c) k^{th} Moment of X.

$$E[X^k] = (-j)^k \frac{d^k \Phi_X(\omega)}{d\omega^k}\bigg|_{\omega=0}$$

$$\Phi_X(\omega) = \sum_{k=0}^{\infty} \frac{1}{k!} \left(\frac{d^k \Phi_X(\omega)}{d\omega^k}\bigg|_{\omega=0} \right) \omega^k = \sum_{m=0}^{\infty} (-1)^m b^{2m} \omega^{2m}$$

Since there are no odd powers in the Taylor series expansion of $\Phi_X(\omega)$, all odd moments of X are zero. For even values of k, we note from the above expressions that

$$\frac{1}{(2k)!} \left(\frac{d^{2k} \Phi_X(\omega)}{d\omega^{2k}}\bigg|_{\omega=0} \right) = (-1)^k b^{2k}$$

$$\Rightarrow \frac{d^k \Phi_X(\omega)}{d\omega^k}\bigg|_{\omega=0} = (k!) j^k b^k$$

$$\Rightarrow E[X^k] = (k!) b^k.$$

Problem 5.22

$$\begin{aligned} X &\sim N(1,4) \\ Y &\sim N(-2,9) \\ Z &= 2X - 3Y - 5 \\ \rho_{X,Y} &= \frac{1}{3} \\ \Rightarrow Cov(X,Y) &= \frac{3 \times 2}{3} = 2 \end{aligned}$$

We will make use of the fact that a linear transformation of the Gaussian Random variables results in a Gaussian Random variable.

$$E[Z] = E[2X - 3Y - 5] = 2(1) - 3(-2) - 5 = 3$$

$$\begin{aligned} Var(Z) &= E[(Z-3)^2] = E[(2X - 3Y - 8)^2] \\ &= E[(2X - 2 - 3Y - 6)^2] \\ &= E[4(X-1)^2 + 9(Y+2)^2 - 12(X-1)(Y+2)] \\ &= 4Var(X) + 9Var(Y) - 12Cov(X,Y) \\ &= 4(4) + 9(9) - 12(2) = 73 \end{aligned}$$

Hence Z is a Gaussian distributed as follows

$$Z \sim N(3, 73)$$

Problem 5.28

Since the transformation is linear and X and Y are jointly Gaussian, U and V will be jointly Gaussian with

$$\begin{aligned}
E[U] &= E[X]\cos(\theta) - E[Y]\sin(\theta) = 0 \\
E[V] &= E[X]\sin(\theta) + E[Y]\cos(\theta) = 0 \\
Var(U) &= E[U^2] = E[X^2]\cos^2(\theta) + E[Y^2]\sin^2(\theta) - 2E[XY]\cos(\theta)\sin(\theta) \\
&= \cos^2(\theta) + \sin^2(\theta) = 1 \\
Var(V) &= E[V^2] = E[X^2]\sin^2(\theta) + E[Y^2]\cos^2(\theta) + 2E[XY]\cos(\theta)\sin(\theta) \\
&= \cos^2(\theta) + \sin^2(\theta) = 1 \\
Cov(U,V) &= E[UV] = E[X^2]\cos(\theta)\sin(\theta) - E[Y^2]\cos(\theta)\sin(\theta) + E[XY](\cos^2(\theta) - \sin^2(\theta)) \\
&= \cos(\theta)\sin(\theta) - \cos(\theta)\sin(\theta) = 0
\end{aligned}$$

Hence, U and V are independent standard Normal random variables.

Problem 6.10

Assume the multivariate normal random variables $X=[x_1, x_2, \ldots, x_N]^T$ with mean vector of μ and covariance matrix of $\sum$. If we partition the X to two groups of $X_1=[x_1, x_2, \ldots, x_q]^T$ and $X_2=[x_{q+1}, x_{q+2}, \ldots, x_q]^T$, then we can write:

$$X=\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \text{ with sizes } \begin{bmatrix} q\times 1 \\ (N-q)\times 1 \end{bmatrix}$$

$$\mu=\begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} \text{ with sizes } \begin{bmatrix} q\times 1 \\ (N-q)\times 1 \end{bmatrix}$$

$$\Sigma=\begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix} \text{ with sizes } \begin{bmatrix} q\times q & q\times(N-q) \\ (N-q)\times q & (N-q)\times(N-q) \end{bmatrix}$$

Then the distribution of X_1 conditioned on $X_2=a$ is also multivariate normal $(X_1| X_2=a)\sim N(\mu_C,\sum_C)$, where

$$\mu_C=\mu_1+\Sigma_{12}\Sigma_{22}^{-1}(a-\mu_2)$$
$$\Sigma_C=\Sigma_{11}-\Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$

a) Using the above information for our particular problem

$$X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix},\ \mu=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},\ \Sigma=\sigma^2\begin{bmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{bmatrix}$$

We can define

$$\mu_1=0,\ \mu_2=\begin{bmatrix} 0 \\ 0 \end{bmatrix},\ a=\begin{bmatrix} x_2 \\ x_3 \end{bmatrix}$$

$$\Sigma_{11}=\sigma^2,\ \Sigma_{12}=\sigma^2\left[\rho \quad \rho\right],\ \Sigma_{22}=\sigma^2\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$$

$$\begin{aligned} E[X_1 \mid X_2=x_2, X_3=x_3] &= \mu_{X_1|X_2,X_3}=\mu_1+\Sigma_{12}\Sigma_{22}^{-1}(a-\mu_2) \\ &= 0+\sigma^2\left[\rho \quad \rho\right]\frac{1}{\sigma^2(1-\rho^2)}\begin{bmatrix} 1 & -\rho \\ -\rho & 1 \end{bmatrix}\left(\begin{bmatrix} x_2 \\ x_3 \end{bmatrix}-\begin{bmatrix} 0 \\ 0 \end{bmatrix}\right) \\ &= \frac{\rho}{1+\rho}(x_2+x_3) \end{aligned}$$

b)

$$
\begin{aligned}
E\left[X_1X_2 \mid X_3=x_3\right] &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_1x_2 f_{X_1,X_2|X_3}\left(x_1,x_2\right) dx_1dx_2 \\
&= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_1x_2 f_{X_1|X_2,X_3}\left(x_1\right) f_{X_2|X_3}\left(x_2\right) dx_1dx_2 \\
&= \int_{-\infty}^{+\infty} x_2\left[\int_{-\infty}^{+\infty} x_1 f_{X_1|X_2,X_3}\left(x_1\right) dx_1\right] f_{X_2|X_3}\left(x_2\right) dx_2 \\
&= \int_{-\infty}^{+\infty} x_2 E\left[X_1 \mid X_2=x_3, X_3=x_3\right] f_{X_2|X_3}\left(x_2\right) dx_2 \\
&= E\left[x_2 E\left[X_1 \mid X_2=x_3, X_3=x_3\right] \mid X_3=x_3\right] \\
&= E\left[x_2 \frac{\rho}{1+\rho}\left(x_2+x_3\right) \mid X_3=x_3\right] \\
&= \frac{\rho}{1+\rho} E\left[x_2^2 + x_2x_3 \mid X_3=x_3\right] \\
&= \frac{\rho}{1+\rho} E\left[x_2^2 \mid X_3=x_3\right] + \frac{\rho}{1+\rho} E\left[x_2x_3 \mid X_3=x_3\right]
\end{aligned}
$$

As $\mathrm{E}\left[\mathrm{g}(\mathrm{Y})\,\mathrm{Z}|\mathrm{Y=y}\right]=\mathrm{g}(\mathrm{y})\,\mathrm{E}\left[\mathrm{Z}|\mathrm{Y=y}\right]$, we can write

$$
E\left[X_1X_2 \mid X_3=x_3\right] = \frac{\rho}{1+\rho} E\left[x_2^2 \mid X_3=x_3\right] + \frac{\rho}{1+\rho} x_3 E\left[x_2 \mid X_3=x_3\right]
$$

Again using the information provided in the previous page, we know that for a pair of jointly Gaussian random variables X_2 and X_3, the pdf of X_2 conditioned on X_3 would be a normal distribution by the following properties

$$
\begin{aligned}
X_2 &: \left(\mu_2 + \rho_{X_2X3}\frac{\sigma_{X_2}}{\sigma_{X_3}}\left(x_3-\mu_3\right),\ \ \sigma_{X_2}^2\left(1-\rho_{X_2X3}^2\right)\right) \\
X_2 &: \left(0+\rho\frac{\sigma}{\sigma}\left(x_3-0\right),\ \ \sigma^2\left(1-\rho^2\right)\right) \\
X_2 &: \left(\rho x_3,\ \ \sigma^2\left(1-\rho^2\right)\right)
\end{aligned}
$$

Thus

$$
\begin{aligned}
E\left[X_1X_2 \mid X_3=x_3\right] &= \frac{\rho}{1+\rho}\left[\sigma^2\left(1-\rho^2\right)\right] + \frac{\rho}{1+\rho} x_3\left(\rho x_3\right) \\
&= \sigma^2\rho\left(1-\rho\right) + \frac{\rho^2}{1+\rho} x_3^2
\end{aligned}
$$

c) Since $\mathrm{E}\left[\mathrm{g}(\mathrm{Y})\,Z\right]=\mathrm{E}\left[\mathrm{g}(\mathrm{Y})\,\mathrm{E}\left[\mathrm{Z}|\mathrm{Y}\right]\right]$, we can write

$$
\begin{aligned}
E[X_1X_2X_3] &= E\left[X_3E[X_1X_2 \mid X_3]\right] \\
&= E\left[X_3\left(\sigma^2\rho(1-\rho)+\frac{\rho^2}{1+\rho}X_3^2\right)\right] \\
&= E\left[X_3\sigma^2\rho(1-\rho)\right]+E\left[\frac{\rho^2}{1+\rho}X_3^3\right] \\
&= \sigma^2\rho(1-\rho)\,E[X_3]+\frac{\rho^2}{1+\rho}E\left[X_3^3\right] \\
&= 0
\end{aligned}
$$

The last equality came from this fact that the all odd moments of a zero mean Gaussian distribution are zero.