MVE136 Random Signals Analysis

Written exam Thursday 17 January 2013 2 – 6 pm

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AIDS: Beta <u>or</u> 2 sheets (4 pages) of hand-written notes (computer print-outs and/or xerox-copies are not allowed), but not both these aids.

GRADES: 12 (40%), 18 (60%) and 24 (80%) points for grade 3, 4 and 5, respectively.

GOOD LUCK!

Task 1. Let $X(t)$ and $Y(t)$ be independent Poisson processes, both with rates 1. Define $Z(t) = X(t) + Y(t)$. Find $E[X(1) \,|\, Z(2) = 2]$. **(5 points)**

Task 2. Let $X(t)$ be a Poisson process with mean function $\mu_X(t) = \lambda\, t$ and autocovariance function $C_{XX}(s,t) = \lambda \min(s,t)$. Show that the process $Y(t) = (X(t) - \lambda\, t)/\sqrt{t}$ is so called group-WSS with respect to multiplication on $(0,\infty)$, which is to say that $E[Y(ht)] = E[Y(t)]$ and $E[Y(hs)Y(ht)] = E[Y(s)Y(t)]$ for $h, s, t > 0$. **(5 points)**

Task 3. Let $X(t)$ be a continuous-time strict sense stationary Gaussian process with zero mean and autocorrelation function $R_{XX}(\tau) = \mathrm{e}^{-|\tau|}$. Show that the process $Y(t) = \mathrm{e}^{X(t)^2}$ is strict sense stationary but not WSS. **(5 points)**

Task 4. Consider a discrete time Markov chain $X(n)$ with state space E, initial distribution $\pi(0)$ and transition probability matrix P given by

$$E = \{0,1,2,3,4\}, \quad \pi(0) = [0\ 0\ 1\ \ 0\ \ 0] \quad \text{and} \quad P = \begin{bmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 & 0 \\ 1/6 & 1/6 & 1/3 & 1/6 & 1/6 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

respectively. Find $E[X(n)]$. **(5 points)**

Task 5. A WSS discrete-time random process $X(n)$ with PSD $S_{XX}(f)$ is input to two different LTI systems with transfer functions $H_1(f)$ and $H_2(f)$, respectively. Find the cross spectral density $S_{Y_1Y_2}(f)$ between the outputs $Y_1(n)$ and $Y_2(n)$ from the two LTI systems. **(5 points)**

Task 6. Explain what weakness of the periodogram that the modified periodogram tries to remove. What is the difference (/are the differences) between the periodogram and the modified periodogram? **(5 points)**

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Solutions to written exam Thursday 17 January 2013

Task 1. We have $P(X(1)=k\,|\,Z(2)=2) = P(X(1)=k,\, X(2)+Y(2)=2)/P(X(2)+Y(2)=2) = [P(X(1)=k,\, X(2)+Y(2)=2,\, Y(2)=0) + P(X(1)=k,\, X(2)+Y(2)=2,\, Y(2)=1) + P(X(1)=k,\, X(2)+Y(2)=2,\, Y(2)=2)]/[P(X(2)=2,\, Y(2)=0) + P(X(2)=1,\, Y(2)=1) + P(X(2)=0,\, Y(2)=2)] = [P(X(1)=k)\,P(X(2)-X(1)=2-k)\,P(Y(2)=0) + P(X(1)=k)\,P(X(2)-X(1)=1-k)\,P(Y(2)=1) + P(X(1)=k)\,P(X(2)-X(1)=-k)\,P(Y(2)=2)]/[\frac{1}{2}\,\mathrm{e}^{-2}+\mathrm{e}^{-2}+\frac{1}{2}\,\mathrm{e}^{-2}] = \frac{1}{2}\,\mathrm{e}\,[P(X(1)=k)\,P(X(1)=2-k) + P(X(1)=k)\,P(X(1)=1-k) + P(X(1)=k)\,P(X(1)=-k)\,\frac{1}{2}] = \mathrm{e}^{-1}\,[\delta(k) + \delta(k-1) + \frac{1}{4}\,\delta(k-2)]$, so that $E[X(1)\,|\,Z(2)=2] = \mathrm{e}^{-1}\,[1\cdot 0 + 1\cdot 1 + \frac{1}{4}\cdot 2] = \frac{3}{2}\,\mathrm{e}^{-1}$.

Task 2. We have $E[Y(ht)] = E[(X(ht)-\lambda\, ht)/\sqrt{ht}] = (\mu_X(ht)-\lambda\, ht)/\sqrt{ht} = 0 = E[Y(t)]$ (where the last equality follows from taking $h=1$) and $E[Y(hs)Y(ht)] = C_{XX}(hs,ht)/\sqrt{h^2 st} = \lambda\,\min(hs,ht)/\sqrt{h^2 st} = \lambda\,\min(\sqrt{s/t}\,,\sqrt{t/s}\,) = E[Y(s)Y(t)]$.

Task 3. As $X(t)$ is strict sense stationary we have $P(Y(t_1+h)\le x_1,\ldots,Y(t_n+h)\le x_n) = 0 = P(Y(t_1)\le x_1,\ldots,Y(t_n)\le x_n)$ if $\min(x_1,\ldots,x_n)\le 0$ while $P(Y(t_1+h)\le x_1,\ldots,Y(t_n+h)\le x_n) = P(-\sqrt{\ln(x_1)}\le X(t_1+h)\le\sqrt{\ln(x_1)}\,,\ldots,-\sqrt{\ln(x_n)}\le X(t_n+h)\le\sqrt{\ln(x_n)}\,) = P(-\sqrt{\ln(x_1)}\le X(t_1)\le\sqrt{\ln(x_1)}\,,\ldots,-\sqrt{\ln(x_n)}\le X(t_n)\le\sqrt{\ln(x_n)}\,) = P(Y(t_1)\le x_1,\ldots,Y(t_n)\le x_n)$ otherwise, so that also $Y(t)$ is strict sense stationary. However, as $\mu_Y(t) = E[Y(t)] = E[\mathrm{e}^{X(t)^2}] = \int_{-\infty}^{\infty}\mathrm{e}^{x^2}\,\frac{1}{\sqrt{2\pi}}\,\mathrm{e}^{-x^2/2}\,dx = \infty$ does not exist $Y(t)$ is not WSS.

Task 4. $E[X(n)] = (1/3)^n\,2 + \frac{1}{2}\,(1-(1/3)^n)\,\frac{1}{2} + \frac{1}{2}\,(1-(1/3)^n)\,\frac{7}{2} = 2$.

Task 5. We have $S_{Y_1Y_2}(f) = \sum_{\tau=-\infty}^{\infty}\mathrm{e}^{-j2\pi f\tau}\,E[\sum_{k=-\infty}^{\infty}h_1(k)X(n-k)\sum_{\ell=-\infty}^{\infty}h_2(\ell)\,X(n+\tau-\ell)] = \sum_{k=-\infty}^{\infty}\sum_{\ell=-\infty}^{\infty}\mathrm{e}^{-j2\pi f(\ell-k)}h_1(k)h_2(\ell)\sum_{\tau=-\infty}^{\infty}\mathrm{e}^{-j2\pi f(\tau-\ell+k)}R_{XX}(\tau-\ell+k) = \overline{H_1(f)}H_2(f)S_{XX}(f)$.

Task 6. The periodogram has limited resolution whenever the sampled (observed) sequence has finite length, N. Perhaps the most serious problem is that weak frequency components can be hidden/masked in the sidelobes of a strong frequency component. The modified periodogram tries to limit this "frequency masking". Another way to put it, is that the periodogram has a bias for finite N and that the modified periodogram tries to reduce the bias.

The difference between the periodogram and the modified periodogram can be explained as follows. The original periodogram is the DTFT

$$\hat{P}_{per}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} w_R[n]x[n]\,\mathrm{e}^{j\omega n},$$

where

$$w_R[n] = \begin{cases} 1 & \text{for } 0 \le n \le N-1 \\ 0 & \text{otherwise.} \end{cases}$$

This rectangular window is introduced as one way to describe that we have only observed $x[n]$ for $n = 0, 1, \ldots, N-1$. One problem with $w_R[n]$ is that its Fourier transform, $W_R(e^{j\omega})$, has very large sidelobes which give rise to the frequency masking mentioned above. The modified periodogram reduces the sidelobes by replacing $w_R[n]$ with a different window function that has lower sidelobes (but unfortunately the alternative windows always have wider mainlobes).