MVE136 Random Signals Analysis

Written exam Monday 19 August 2013 2 – 6 pm

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AIDS: Beta or 2 sheets (4 pages) of hand-written notes (computer print-outs and/or xerox-copies are not allowed), but not both these aids.

GRADES: 12 (40%), 18 (60%) and 24 (80%) points for grade 3, 4 and 5, respectively.

GOOD LUCK!

Task 1. Let X and Y be zero-mean, unit variance Gaussian random variables with correlation coefficient ρ. Suppose we form two new random variables U and V using a linear trasnformation as

$$U = a\,X + b\,Y \quad \text{and} \quad V = c\,X + d\,Y,$$

where a, b, c and d are real (non-random) coefficients/constants. Find constraints on a, b, c and d such that U and V are independent. **(5 points)**

Task 2. Find the probability $Pr(X(1) + X(2) > 3)$ for a Poisson process with rate 1. **(5 points)**

Task 3. Consider a discrete time Markov chain $X(n)$ with state space E and transition probability matrix P given by

$$E = \{0, 1\} \qquad \text{and} \qquad P = \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix},$$

respectively. What initial distribution $\pi(0)$ of the chain will give a distribution $\pi(2)$ of the value of the chain $X(2)$ at time $n = 2$ given by $\pi(2) = [1/3 \;\; 2/3]$? **(5 points)**

Task 4. The PSD $S_{XX}(f)$ of a continuous time WSS process $X(t)$ has the properties to be real and symmetric (=even). Prove one of these properties. **(5 points)**

Task 5. Give an example of a continuous time impulse response function h that has the property that when a continuous time white noise process $N(t)$ with PSD $S_{NN}(f) = N_0/2$ is input to an LTI system with this impulse response, then the output process $Y(t)$ from the system has PSD $S_{YY}(f)$ that is decreasing for $f \geq 0$ and satisfies $S_{YY}(f_0) = S_{YY}(0)/2 = N_0/4$ for a certain frequency $f_0 > 0$. (In other words the LTI system is a lowpass filter with 3dB bandwith f_0.) **(5 points)**

Task 6. Suppose that we observe

$$x[n] = d[n] + w[n],$$

where both $d[n]$ and $w[n]$ are wide sense stationary signals and

$$\mathrm{E}\{d[n-k]\, w[n]\} = 0 \quad \text{for all } k \text{ and } n.$$

Further assume that $r_d[0] = 1$ and $r_d[1] = 0.5$ whereas $r_w[n] = \delta[n]$.

Your task is to design a causal FIR-Wiener filter of length two:

$$\hat{d}[n] = h[0]\, x[n] + h[1]\, x[n-1].$$

Use the Wiener-Hopf equations to compute the optimal filter coefficients $h[0]$ and $h[1]$, i.e., the coefficients that minimize the mean squared error $\mathrm{E}\{(d[n] - \hat{d}[n])^2\}$. **(5 points)**

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Solutions to written exam Monday 19 August 2013

Task 1. This is Exercise 5.29 in the 2004 edition of Miller and Childers's (=Exercise 5.30 in the 2012 edition). The solution is available at the URL

`www.math.chalmers.se/Stat/Grundutb/CTH/mve135/1011/Exercises/HomeExercise4.pdf`

Task 2. As $X(1)+X(2) = (X(2)-X(1)) + 2\,X(1)$ where $X(2)-X(1)$ and $X(1)$ are independent Po(1)-distributed we have $Pr(X(1)+X(2) > 3) = Pr((X(2)-X(1)) + 2\,X(1) > 3) = Pr(X(1) \geq 2) + Pr(X(1)=1,\, X(2)-X(1) > 1) + Pr(X(1)=0,\, X(2)-X(1) > 3) = (1-\mathrm{e}^{-1}-\mathrm{e}^{-1}) + \mathrm{e}^{-1}\,(1-\mathrm{e}^{-1}-\mathrm{e}^{-1}) + \mathrm{e}^{-1}\,(1-\mathrm{e}^{-1}-\mathrm{e}^{-1}-\frac{1}{2}\mathrm{e}^{-1}-\frac{1}{6}\mathrm{e}^{-1})$.

Task 3. Writing $\pi(0) = [p\ \ 1-p]$, we have

$$[1/3\ \ 2/3] = \pi(2) = \pi(0)\,P^2 = [p\ \ 1-p]\begin{bmatrix} 3/8 & 5/8 \\ 5/16 & 11/16 \end{bmatrix} = [\frac{5}{16}+\frac{p}{16}\ \ \frac{11}{16}-\frac{p}{16}] \Leftrightarrow p = \frac{1}{3}.$$

Task 4. As the autocorrelation function $R_{XX}(\tau)$ is symmetric we have $S_{XX}(-f) = \int_{-\infty}^{\infty} R_{XX}(\tau)\,\mathrm{e}^{-j2\pi(-f)\tau}\,d\tau = \int_{-\infty}^{\infty} R_{XX}(\tau)\,\mathrm{e}^{-j2\pi f(-\tau)}\,d\tau = \int_{-\infty}^{\infty} R_{XX}(-\hat{\tau})\,\mathrm{e}^{-j2\pi f\hat{\tau}}\,d\hat{\tau} = \int_{-\infty}^{\infty} R_{XX}(\hat{\tau})\,\mathrm{e}^{-j2\pi f\hat{\tau}}\,d\hat{\tau} = S_{XX}(f)$ and $\overline{S_{XX}(f)} = \overline{\int_{-\infty}^{\infty} R_{XX}(\tau)\,\mathrm{e}^{-j2\pi f\tau}\,d\tau} = \int_{-\infty}^{\infty} R_{XX}(\tau)\,\mathrm{e}^{j2\pi f\tau}\,d\tau = [\text{see above}] = S_{XX}(f)$.

Task 5. If we take $h(t) = \exp(-t/t_0)\,u(t)/t_0$ then we have $H(f) = F[h(t)] = 1/(1+j2\pi f t_0)$, so that $S_{YY}(f) = |H(f)|^2 S_{NN}(f) = (N_0/2)/(1+4\pi^2 f^2 t_0^2)$ satisfies the imposed requirements when $t_0 = 1/(2\pi f_0)$.

Task 6. For completeness, this solution will contain a derivation of the Wiener-Hopf (WH) equations. Let us denote the estimation error as $e[n] = d[n] - \hat{d}[n]$. We know that the optimal filter is such that

$$\mathrm{E}\{e[n]\,x[n-k]\} = 0 \quad \text{for} \ \ k = 0, 1.$$

We obtain the WH-equations by plugging in the expression for $e[n]$ in the above equation:

$$\mathrm{E}\{(d[n]-h[0]\,x[n]-h[1]\,x[n-1])\,x[n-k]\} = r_{dx}[k] - h[0]\,r_x[k] - h[1]\,r_x[k-1] = 0$$

for $k = 0, 1$. We can express these equations in matrix form as

$$\begin{bmatrix} r_x[0] & r_x[1] \\ r_x[1] & r_x[0] \end{bmatrix} \begin{bmatrix} h[0] \\ h[1] \end{bmatrix} = \begin{bmatrix} r_{dx}[0] \\ r_{dx}[1] \end{bmatrix}.$$

Here we have

$$r_{dx}[k] = \mathrm{E}\{d[n]\, x[n-k]\} = \mathrm{E}\{d[n]\, (d[n-k] + w[n-k])\} = r_d[k]$$

since $d[n]$ and $w[n-k]$ are uncorrelated. Similarly,

$$r_x[k] = \mathrm{E}\{(d[n] + w[n])\, (d[n-k] + w[n-k])\} = r_d[k] + r_w[k].$$

Inserting this together with the information given in the problem into the matrix form of the WH-equation we arrive at

$$\begin{bmatrix} 2 & 0.5 \\ 0.5 & 2 \end{bmatrix} \begin{bmatrix} h[0] \\ h[1] \end{bmatrix} = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$$

with solution

$$h[0] = \frac{1 - \frac{1}{8}}{2 - \frac{1}{8}} \approx 0.47 \quad \text{and} \quad h[1] = \frac{8}{60} \approx 0.13,$$

which is the final answer to the problem.