MVE136 Random Signals Analysis

Written exam Tuesday 5 January 2016 2–6 pm

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AIDS: Beta or 2 sheets (=4 pages) of hand-written notes (computer print-outs and/or xerox-copies are not allowed), but not both these aids.

GRADES: 12 (40%), 18 (60%) and 24 (80%) points for grade 3, 4 and 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Task 1. Let $X(t)$ be a continuous time WSS random process with autocorrelation function $R_{XX}(\tau)$. Given a constant $\varepsilon > 0$, form a new process $Y(t)$ as $Y(t) = (X(t+\varepsilon) - X(t))/\varepsilon$ for $t \in \mathbb{R}$. Is $Y(t)$ WSS? **(5 points)**

Task 2. Let $X_1(t), \ldots, X_n(t)$ be independent Poisson processes with arrival rates $\lambda_1, \ldots, \lambda_n > 0$, respectively. Show that $X(t) = X_1(t) + \ldots + X_n(t)$ is also a Poisson process at find its arrival rate λ. **(5 points)**

Task 3. Find the transition matrix P for a Markov chain $X[k]$ with two states and stationary distribution $\pi = [1/2 \;\; 1/2]$. **(5 points)**

Task 4. Let $\{W(t)\}_{t \geq 0}$ be a random process with autocorrelation function $R_{WW}(s,t) = \min\{s,t\}$. Form a new process $Y(t)$ as $Y(t) = \int_0^t W(u)\, du$ for $t \geq 0$. Find the autocorrelation function $R_{YY}(s,t)$ for $0 \leq s \leq t$. **(5 points)**

Task 5. Give a proof of the formula for the Wiener filter according to Section 11.6 in the book by Miller & Childers for the case when the noise process $N(t)$ is zero-mean and independent of the input signal (the signal to be estimated) $Z(t)$. **(5 points)**

Task 6. We are interested in estimating the spectrum of a stationary stochastic process, but unfortunately we have only been able to collect six data points $x[0] = -1.4$, $x[1] = 0.1$, $x[2] = -0.7$, $x[3] = -1$, $x[4] = 0.3$ and $x[5] = 1.5$. Given that we have a small data set, we decide to use a simple AR(1) model. Use the data to estimate the AR model and its spectrum (for simplicity you can assume that the variance of the input noise is $\sigma_e^2 = 1$). We would also like you to sketch the spectrum. **(5 points)**

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Solutions to written exam 5 January 2016

Task 1. Elementary calculations show that $\mu_Y(t) = 0$ and $R_{YY}(t, t+\tau) = (2R_{XX}(\tau) - R_{XX}(\tau-\varepsilon) - R_{XX}(\tau+\varepsilon))/\varepsilon^2$, neither of which depend on t, so that $Y(t)$ is WSS.

Task 2. Just check that the defining properties of a Poisson process at the beginning of Section 8.6 in the book by Miller & Childers remain valid for the process $X(t)$ with arrival rate $\lambda = \lambda_1 + \ldots + \lambda_n$.

Task 3. The equation $\pi = \pi P$ with $\pi = [1/2 \;\; 1/2]$ gives

$$[1/2 \;\; 1/2] = [1/2 \;\; 1/2] \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} \Leftrightarrow [1/2 \;\; 1/2] = [(1-a+b)/2 \;\; (1+a-b)/2] \Leftrightarrow a = b.$$

Task 4. $R_{YY}(s,t) = E\big[(\int_0^s W(u)\,du)\,(\int_0^t W(v)\,dv)\big] = \int_{u=0}^{u=s}\int_{v=0}^{v=t} E[W(u)W(v)]\,dudv = \int_{u=0}^{u=s}\int_{v=0}^{v=s} \min\{u,v\}\,dudv + \int_{u=0}^{u=s}\int_{v=s}^{v=t} \min\{u,v\}\,dudv = 2\int_{u=0}^{u=s}\int_{v=0}^{v=u} v\,dvdu + \int_{u=0}^{u=s}\int_{v=s}^{v=t} u\,dvdu = 2\int_0^s u^2/2\,du + \int_0^s (t-s)\,u\,du = \ldots = ts^2/2 - s^3/6.$

Task 5. See Section 11.6 in the book by Miller & Childers.

Task 6. Let us follow the standard strategy:

1. use the data to estimate the autocorrelation function $r_x[k]$,
2. estimate the AR parameter a_1 from the autocorrelation function, and
3. compute the spectrum, $P_x(\mathrm{e}^{j\omega})$ once we have found the AR model.

To learn more about for which values of k that we need to estimate $r_x[k]$ we can first look at how we intend to estimate the AR parameter. The AR(1) difference equation is

$$x[n] + a_1\,x[n-1] = e[n] \tag{1}$$

and we can derive the Yule-Walker equations by multiplying both sides of (1) by $x[n-k]$ for $k \geq 0$ and take expectations. We then get

$$r_x[k] + a_1\,r_x[k-1] = E\big[(x[n] + a_1\,x[n-1])\,x[n-k]\big] = E[e[n]x[n-k]] = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k > 0 \end{cases}.$$

Considering that we only have one parameter to estimate it is sufficient to estimate $r_x[0]$ and $r_x[1]$ since that would enable us to use $r_x[0] + a_1 r_x[1] = 1$ to find a_1. (Recall that $r_x[k] = r_x[-k]$ for wide sense stationary processes).

To estimate the autocorrelation function we can use the estimator

$$\hat{r}_x[k] = \frac{1}{N} \sum_{n=k}^{N-1} x[n]\, x[n-k],$$

under the assumption that $0 \leq k \leq N-1$. From this we obtain

$$\begin{cases} \hat{r}_x[0] = [(-1.4)^2 + 0.1^2 + (-0.7)^2 + (-1)^2 + 0.3^2 + 1.5^2]/6 \approx 0.97 \\ \hat{r}_x[1] = [(-1.4)\cdot 0.1 + 0.1\cdot(-0.7) + (-0.7)\cdot(-1) + (-1)\cdot 0.3 + 0.3\cdot 1.5]/6 \approx 0.11 \end{cases}.$$

We can now solve for the AR parameter

$$a_1 = \frac{1 - r_x[0]}{r_x[1]} \approx \frac{1 - \hat{r}_x[0]}{\hat{r}_x[1]} \approx 0.31.$$

Assuming that $x[n]$ is an AR(1) model, see (1), it can described as white noise propagated through a linear and time invariant filter with the transfer function

$$H(z) = \frac{1}{1 + a_1\, z^{-1}}.$$

To sketch the spectrum we note that the transfer function has a pole at $z = -a_1 \approx -0.31$. This implies that the region of convergence (ROC) of this z-transform includes the unit circle (the ROC is outside the pole for causal filters), which means that the frequency response $H(\mathrm{e}^{j\omega})$ exists. Further, we know that $|H(\mathrm{e}^{j\omega})|$ is large close to the pole and small far from the pole. The spectrum is conveniently expressed as

$$P_x(\mathrm{e}^{j\omega}) = \sigma_e^2\, |H(\mathrm{e}^{j\omega})|^2 = |H(\mathrm{e}^{j\omega})|^2 = \frac{1}{|1 + a_1\, \mathrm{e}^{-j\omega}|^2},$$

where we assume $a_1 \approx 0.31$. An illustration of this spectrum is given in Figure 1 below.

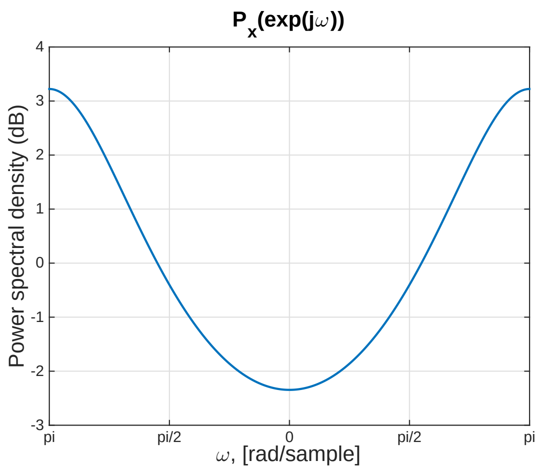

Figure 1: An illustration of our approximation to the power spectral density $P_x(e^{j\omega})$.