MVE136 Random Signals Analysis

Written exam Wednesday 29 October 2014 2–6 pm

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AIDS: Beta or 2 sheets (4 pages) of hand-written notes (computer print-outs and/or xerox-copies are not allowed), but not both these aids.

GRADES: 12 (40%), 18 (60%) and 24 (80%) points for grade 3, 4 and 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Task 1. Explain why one of the functions $S_1(f) = \cos(f)$, $S_2(f) = \mathrm{e}^{-|f|}$ and $S_3(f) = 1/(1+f^2)$ is not the PSD of a continuous time WSS random process. **(5 points)**

Task 2. A WSS continuous time white noise process $N(t)$ with PSD $S_{NN}(f) = N_0/2$ is input to an LTI system with impulse response $h(t) = \sqrt{8}/(1+(2\pi t)^2)$. Show that the autocorrelation function of the output $Y(t)$ is $R_{YY}(\tau) = N_0/(1+(\pi\tau)^2)$. **(5 points)**

Task 3. Let $X(t)$ be a WSS random process and define a new random process $Y(t)$ by setting $Y(t) = X(-t)$. Show that $Y(t)$ is a WSS random process. **(5 points)**

Task 4. Let X_k be a Markov chain that has a stationary distribution π and that has initial distribution $\pi(0) = \pi$. Show that $E[X_k X_{k+1}] = \sum_i \sum_j ij\, p_{i,j}\, \pi_i$. **(5 points)**

Task 5. Let $X(t)$ be a WSS continuous time zero-mean Gaussian random process with autocorrelation function $R_{XX}(\tau) = \mathrm{e}^{-|\tau|}$. Find the PDF of the random variable $X(0) + \int_0^t X(r)\, dr$ for $t > 0$. [HINT: $\int_0^t \int_0^t \mathrm{e}^{-|r-s|}\, dr ds = 2\,(t-(1-\mathrm{e}^{-t}))$.] **(5 points)**

Task 6. We have collected four measurements

$$x[0] = 0.3, \quad x[1] = -0.2, \quad x[2] = 0.1 \quad \text{and} \quad x[3] = -0.3,$$

of a discrete time WSS random process $\{x[n]\}_{n=-\infty}^{\infty}$ and we wish to find a mathematical model for the measured process. We select to use a simple AR(1)-model $x[n] + a_1\, x[n-1] = e[n]$ for $n \in \mathbb{Z}$, where $\{e[n]\}_{n=-\infty}^{\infty}$ is discrete time white noise: Your task is to estimate the parameter a_1 and the white noise variance $\sigma_e^2 = \mathbf{E}\{e[n]^2\}$. **(5 points)**

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Solutions to written exam 29 October 2014

Task 1. As $S_1(f)$ is not a nonnegative function it cannot be a PSD, cf. Equation 10.12b in Miller & Childers.

Task 2. As $S_{YY}(f) = |H(f)|^2 S_{NN}(f) = |(Fh)(f)|^2 S_{NN}(f) = |\sqrt{2}\,\mathrm{e}^{-|f|}|^2 N_0/2 = N_0\,\mathrm{e}^{-2|f|}$ we see that $R_{YY}(\tau) = (F^{-1}S_{YY})(\tau) = N_0/(1+(\pi\tau)^2)$ as claimed (where we made use twice of the fourth transform pair in Table E.1 in Miller & Childers).

Task 3. Clearly $\mu_Y(t) = \mu_X(-t)$ and $R_{YY}(t,t+\tau) = R_{XX}(-t,-(t+\tau)) = R_{XX}(-t, -t-\tau) = R_{XX}(-\tau) = R_{XX}(\tau)$ is independent of t when $X(t)$ is WSS.

Task 4. $E[X_k X_{k+1}] = \sum_i E[X_k X_{k+1}|X_k = i]\, P_{X_k}(i) = \sum_i E[i\, X_{k+1}|X_k = i]\, \pi(k)_i = \sum_i i\, E[X_{k+1}|X_k = i]\, \pi_i = \sum_i i \sum_j j\, \Pr(X_{k+1} = j|X_k = i)\, \pi_i = \sum_i \sum_j ij\, p_{i,j}\, \pi_i$.

Task 5. The random variable is Gaussian with mean $E[X(0) + \int_0^t X(r)\,dr] = E[X(0)] + \int_0^t E[X(r)]\,dr = 0$ and variance $E[X(0)^2] + 2\,E[X(0)\int_0^t X(r)\,dr] + E[(\int_0^t X(r)\,dr)^2] = 1 + 2\int_0^t E[X(0)X(r)]\,dr + \int_0^t\int_0^t E[X(r)X(s)]\,drds = 1 + 2\int_0^t \mathrm{e}^{-|0-r|}\,dr + \int_0^t\int_0^t \mathrm{e}^{-|r-s|}\,drds = 1 + 2(1-\mathrm{e}^{-t}) + 2(t-(1-\mathrm{e}^{-t})) = 1+2t$ so the PDF is that of a zero-mean Gaussian random variable with variance $1+2t$.

Task 6. As the autocorrelation function $r_x[k]$ of the AR(1)-process satisfies the Yule-Walker equations

$$a_1\, r_x[0] + r_x[1] = 0 \quad \text{and} \quad r_x[0] + a_1\, r_x[1] = \sigma_e^2,$$

we obtain estimates $\hat{a}_1$ and $\hat{\sigma}_e^2$ of the parameters a_1 and σ_e^2 by solving the equations

$$\hat{a}_1\, \hat{r}_x[0] + \hat{r}_x[1] = 0 \quad \text{and} \quad \hat{r}_x[0] + \hat{a}_1\, \hat{r}_x[1] = \hat{\sigma}_e^2,$$

where

$$\hat{r}_x[0] = \frac{1}{4}\sum_{n=0}^{3} x[n]^2 = \ldots = 0.0575 \quad \text{and} \quad \hat{r}_x[1] = \frac{1}{4}\sum_{n=0}^{2} x[n]\,x[n+1] = \ldots = -0.0275$$

are estimated values of the autocorrelation function. This gives $\hat{a}_1 = 0.0275/0.0575$ and $\hat{\sigma}_e^2 = 0.0575 - 0.0275^2/0.0575$.