MVE136 Random Signals Analysis

Written exam Wednesday 28 October 2015 2–6 pm

Teacher and Jour: Patrik Albin, telephone 070 6945709.

Aids: Beta or 2 sheets (=4 pages) of hand-written notes (computer print-outs and/or xerox-copies are not allowed), but not both these aids.

Grades: 12 (40%), 18 (60%) and 24 (80%) points for grade 3, 4 and 5, respectively.

Motivations: All answers/solutions must be motivated. Good luck!

Task 1. Calculate the probability $P(X(1)=1,\ X(4)=4\,|\,X(2)=2,\ X(3)=3,\ X(5)=5)$ for a Poisson process $X(t)$ with arrival rate $\lambda>0$. **(5 points)**

Task 2. A continuous time random process $X(t)$, $t\geq 0$, is called H-selfsimilar if the random variables $(X(\lambda t_1),\ldots,X(\lambda t_n))$ and $(\lambda^H X(t_1),\ldots,\lambda^H X(t_n))$ have the same PDF for each choice of $\lambda>0$, $n\in\mathbb{N}$ and $t_1,\ldots,t_n\geq 0$. Show that a Gaussian process $X(t)$, $t\geq 0$, is H-selfsimilar if and only if its mean and autocovariance functions satisfy $\mu_X(\lambda t)=\lambda^H\mu_X(t)$ and $C_{XX}(\lambda t_1,\lambda t_2)=\lambda^{2H}C_{XX}(t_1,t_2)$, respectively. **(5 points)**

Task 3. Two students play a game consisting of tossing two dice. At each game the students bet one dollar each, if the sum of the two dice is less than 7 student A collects the two dollars in the pot, if the sum is greater than 7 student B collects the pot, while if the sum is 7 the students with the fewest dollars collects the pot (with the pot being shared if the students have the same amounts of dollars). The game continues until one student is bankrupt. Each students have a starting amount of 3 dollars. Find the transition matrix P for the Markov chain $X[k]$ indicating the amount of dollars student A posesses after k games. **(5 points)**

Task 4. Let $N(t)$, $t\in\mathbb{R}$, be a continuous time WSS white noise process with zero-mean, constant PSD $S_{NN}(f)=N_0/2$ and autocorrelation function $R_{NN}(\tau)=(N_0/2)\,\delta(\tau)$. Find the autocorrelation function $R_{WW}(t_1,t_2)$ for the integrated white noise process $W(t)=\int_0^t N(r)\,dr$, $t\geq 0$. **(5 points)**

Task 5. A continuous time deterministic (=non-random) signal $s(t)$ is transmitted on a noisy channel so that the noise disturbed signal $X(t)=s(t)+N(t)$ is recived, where $N(t)$ is a (random) zero-mean white noise process with PSD $S_{NN}(f)=N_0/2$. The task for the electrical engineer is to process the recived signal $X(t)$ through an LTI system with a suitably choosen impulse response $h(t)$ so that the signal to noise ratio $\mathrm{SNR}(t) = E[((s * h)(t))^2]/E[((N * h)(t))^2]$ is maximized at time $t = t_0$.

(a) Show that $\mathrm{SNR}(t) = (2/N_0)[\int_{-\infty}^{\infty} h(u)\, s(t-u)\, du]^2/[\int_{-\infty}^{\infty} h(u)^2\, du]$. **(2.5 points)**

(b) According to Section 11.5 in the book by Miller & Childers the impulse response $h(t)$ that maximizes $\mathrm{SNR}(t_0)$ is given by $h(t) = s(t_0-t)$: Prove this fact! **(2.5 points)**

Task 6. Prediction is an important tool in many contexts, including decision theory, planning and control. Suppose that we wish to predict $x[n+1]$ given $x[n]$ and $x[n-1]$, where $x[n]$ is an MA(2)-process

$$x[n] = e[n] + 0.5\, e[n-1] + 0.2\, e[n-2] \quad \text{for} \ \ n \in \mathbb{N}.$$

The input noise $e[n]$ is assumed to be a wide sense stationary zero mean white noise process with variance $E[e[n]^2] = 1$ and autocorrelation function $r_e[k] = E[e[n]e[n+k]] = 0$ for $k \neq 0$. We will further assume that we seek a linear estimator

$$\hat{x}[n+1] = h_0\, x[n] + h_1\, x[n-1].$$

Find the coefficients h_0 and h_1 that minimize $E[(x[n+1] - \hat{x}[n+1])^2]$. **(5 points)**

MVE136 Random Signals Analysis

Solutions to written exam 28 October 2015

Task 1. $P(X(1)=1,\, X(4)=4 \,|\, X(2)=2,\, X(3)=3,\, X(5)=5)$

$$= \frac{P(X(1)=1,\, X(4)=4,\, X(2)=2,\, X(3)=3,\, X(5)=5)}{P(X(2)=2,\, X(3)=3,\, X(5)=5)}$$

$$= \frac{P(X(1)=1,\, X(2)-X(1)=1,\, X(3)-X(2)=1,\, X(4)-X(3)=1,\, X(5)-X(4)=1)}{P(X(2)=2,\, X(3)-X(2)=1,\, X(5)-X(3)=2)}$$

$$= \frac{[P(X(1)=1)]^5}{P(X(2)=2)\, P(X(2)=1)\, P(X(2)=2)}$$

$$= \frac{[P(X(1)=1)]^4}{[P(X(2)=2)]^2}$$

$$= \frac{[\lambda^1/((1!)\cdot \mathrm{e}^{\lambda})]^4}{[(2\lambda)^2/((2!)\cdot \mathrm{e}^{2\lambda})]^2} = \frac{1}{4}.$$

Task 2. As a Gaussian process $X(t)$, $t \geq 0$, is fully determined by its mean and autocovariance functions it follows that $X(t)$ is H-selfsimilar if and only if the Gaussian processes $X(\lambda t)$, $t \geq 0$, and $\lambda^H X(t)$, $t \geq 0$, have the same mean and autocovariance functions. However, these functions for these processes are given by $\mu_X(\lambda t)$ together with $C_{XX}(\lambda t_1, \lambda t_2)$ and $\lambda^H \mu_X(t)$ together with $\lambda^{2H} C_{XX}(t_1, t_2)$, resepectively.

Task 3.

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 5/12 & 0 & 7/12 & 0 & 0 & 0 & 0 \\ 0 & 5/12 & 0 & 7/12 & 0 & 0 & 0 \\ 0 & 0 & 5/12 & 1/6 & 5/12 & 0 & 0 \\ 0 & 0 & 0 & 7/12 & 0 & 5/12 & 0 \\ 0 & 0 & 0 & 0 & 7/12 & 0 & 5/12 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Task 4. We have $R_{WW}(t_1, t_2) = E[(\int_0^{t_1} N(r)\, dr)\, (\int_0^{t_2} N(s)\, ds)] = \int_0^{t_1} \int_0^{t_2} E[N(r)N(s)]$ $drds = \int_0^{t_1} \int_0^{t_2} (N_0/2)\, \delta(s-r)\, drds = \int_0^{\min(t_1,t_2)} (N_0/2)\, dr = (N_0/2)\, \min(t_1, t_2)$.

Task 5. See Section 11.5 in the book by Miller & Childers.

Task 6. Let us first formulate this problem in terms of our standard notation for Wiener filtering: Our quantity of interest is usually denoted $d[n]$ whereas our estimator is denoted $\hat{d}[n]$. In this problem, we have $d[n] = x[n+1]$ and $\hat{d}[n] = h_0\, x[n] + h_1\, x[n-1]$. In terms of this notation, the Wiener-Hopf equations can be expressed as

$$\begin{cases} h_0\, r_x[0] + h_1\, r_x[1] = r_{dx}[0] \\ h_0\, r_x[1] + h_1\, r_x[0] = r_{dx}[1] \end{cases},$$

which can also be written on matrix form

$$\begin{bmatrix} r_x[0] & r_x[1] \\ r_x[1] & r_x[0] \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} = \begin{bmatrix} r_{dx}[0] \\ r_{dx}[1] \end{bmatrix}. \tag{1}$$

For this problem, we note that $r_{dx}[0] = E[d[n]x[n]] = E[x[n+1]x[n]] = r_x[1]$ and $r_{dx}[1] = E[d[n]x[n-1]] = E[x[n+1]x[n-1]] = r_x[2]$ so that it is therefore sufficient to compute $r_x[k]$ for $k = 0$, 1 and 2 before we can solve for h_0 and h_1.

Let us try to derive a general expression for $r_x[k]$: Given our expression for $x[n]$ it holds that

$$\begin{aligned} r_x[k] &= E[x[n]x[n-k]] \\ &= E\big[(e[n] + 0.5\, e[n-1] + 0.2\, e[n-2])\,(e[n-k] + 0.5\, e[n-1-k] + 0.2\, e[n-2-k])\big] \\ &= \begin{cases} 0 & \text{if } |k| > 2 \\ 0.2 & \text{if } |k| = 2 \\ 0.5 + 0.2 \cdot 0.5 = 0.6 & \text{if } |k| = 1 \\ 1 + 0.5 \cdot 0.5 + 0.2 \cdot 0.2 = 1.29 & \text{if } k = 0 \end{cases}. \end{aligned} \tag{2}$$

By combining (1) with (2), we obtain the matrix equation

$$\begin{bmatrix} 1.29 & 0.6 \\ 0.6 & 1.29 \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 0.2 \end{bmatrix},$$

from which we can find the solutions $h_0 \approx 0.50$ and $h_1 \approx -0.08$.