MVE136 Random Signals Analysis

Written exam Monday 15 August 2016 2–6 pm

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AIDS: Beta or 2 sheets (=4 pages) of hand-written notes (computer print-outs and/or xerox-copies are not allowed), but not both these aids.

GRADES: 12 (40%), 18 (60%) and 24 (80%) points for grade 3, 4 and 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Task 1. Let $X(t)$ and $Y(t)$ be independent Poisson processes with common arrival rate $\lambda > 0$. Calculate the probability $P(X(1)-Y(1)=0\,|\,X(1)+Y(1)=2)$. **(5 points)**

Task 2. Calculate the mean function $\mu_X(t)$ and the autocorrelation function $R_{XX}(s,t)$ for the random process $X(t)$, $t \geq 0$, given by $X(t) = \mathrm{e}^{-At}$, where A is a unit mean exponentially distributed random variable. **(5 points)**

Task 3. Is it possible for a two state Markov chain not to have a stationary distribution? **(5 points)**

Task 4. Let $X(t)$ be a stationary zero-mean unit variance Gaussian random process and form a new random process $Y(t)$ as $Y(t) = X(t)^2$. Calculate the crosscorrelation function $R_{XY}(s,t)$. [Hint: It can be very useful to make use of the fact that $X(s) - R_{XX}(t-s)X(t)$ and $X(t)$ are independent.] **(5 points)**

Task 5. If a WSS random process $X(t)$ with PSD $S_{XX}(f)$ is input signal to an LTI system with transfer function $H(f)$, then the output signal $Y(t)$ from the system is WSS with PSD $S_{YY}(f) = |H(f)|^2 S_{XX}(f)$: Prove this relation. **(5 points)**

Task 6. Consider a situation where we sample a time continuous wide sense stationary stochastic process $x_a(t)$. Suppose that the sampling frequency F_s is large, say at least 1000 Hz, and that we intend to estimate the power spectral density $P_x(\mathrm{e}^{j\omega})$ using Bartlett's method with non-overlapping periodogram averaging, here denoted $\hat{P}_x(\mathrm{e}^{j\omega})$. If we seek to obtain a frequency resolution less than 30 Hz a and relative variance

$$\frac{\mathrm{Var}[\hat{P}_x(\mathrm{e}^{j\omega})]}{P_x(\mathrm{e}^{j\omega})^2} \lessapprox \frac{1}{10},$$

how many samples do we then need to collect (expressed as a function of F_s)? For how many seconds do we need to collect data? **(5 points)**

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Solutions to written exam 15 August 2016

Task 1. $P(X(1)-Y(1)=0\,|\,X(1)+Y(1)=2)$

$$= \frac{P(X(1)-Y(1)=0,\, X(1)+Y(1)=2)}{P(X(1)+Y(1)=2)}$$

$$= \frac{P(X(1)=Y(1)=1)}{P(X(1)=2,\, Y(1)=0) + P(X(1)=1,\, Y(1)=1) + P(X(1)=0,\, Y(1)=2)}$$

$$= \frac{P(X(1)=1)\, P(Y(1)=1)}{P(X(1)=2)\, P(Y(1)=0) + P(X(1)=1)\, P(Y(1)=1) + P(X(1)=0)\, P(Y(1)=2)}$$

$$= \frac{[\lambda/((1!)\,\mathrm{e}^{\lambda})]^2}{[\lambda^2/((2!)\,\mathrm{e}^{\lambda})]\,[\lambda^0/((0!)\,\mathrm{e}^{\lambda})] + [\lambda/((1!)\,\mathrm{e}^{\lambda})]^2 + [\lambda^0/((0!)\,\mathrm{e}^{\lambda})]\,[\lambda^2/((2!)\,\mathrm{e}^{\lambda})]}$$

$$= \frac{1}{2}.$$

Task 2. $\mu_X(t) = E[X(t)] = E[\mathrm{e}^{-At}] = \int_0^\infty \mathrm{e}^{-at}\,\mathrm{e}^{-a}\,da = 1/(1+t)$ and $R_{XX}(s,t) = E[X(s)X(t)] = E[\mathrm{e}^{-As}\mathrm{e}^{-At}] = \ldots = 1/(1+s+t)$.

Task 3. A stationary distribution π exists if and only if we can solve

$$\begin{cases} [\pi_1\ \ \pi_2] = [\pi_1\ \ \pi_2]\begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} \\ \pi_1+\pi_2 = 1 \end{cases} \Leftrightarrow \begin{cases} a\pi_1 = b\pi_2 \\ \pi_1+\pi_2 = 1 \end{cases} \Leftrightarrow \begin{cases} \begin{cases} \pi_1 = b/(a+b) \\ \pi_2 = a/(a+b) \end{cases} & \text{if } a+b \neq 0 \\ \pi_1+\pi_2 = 1 & \text{if } a+b = 0 \end{cases}.$$

As this always have at least one solution a stationary distribution always exists!

Task 4. $R_{XY}(s,t) = E[X(s)Y(t)] = E[X(s)X(t)^2] = E[(X(s)-R_{XX}(t-s)X(t))\,X(t)^2] + E[R_{XX}(t-s)X(t)^3] = E[X(s)-R_{XX}(t-s)X(t)]\,E[X(t)^2] + R_{XX}(t-s)\,E[X(t)^3] = 0+0 = 0.$

Task 5. See Section 11.1 in the book by Miller & Childers.

Task 6. To solve this problem, it is a good idea to first summarize some of the results that we have learned in this course. For instance, if we have collected a long sequence of data, we know that the relative variance is roughly $1/K$ where K is the number of segments that we split our data sequence in. To obtain the desired relative variance we thus need to select $K \geq 10$.

Another important result is that the resolution is $2\pi/N$ rad/s for the periodogram and $2\pi K/N$ for the averaged periodogram, where $M = N/K$ is the number of samples in the individual segments. To obtain a frequency resolution smaller than 30 Hz when the sampling frequency is F_s we need to have a frequency resolution in discrete time which is smaller than $30\,2\pi/F_s = 60\,\pi/F_s$ rad/sample. The conclusion from this is that N need to be sufficiently large in order to satisfy

$$2\pi K/N \leq 60\,\pi/F_s.$$

From this inequality we get $N \geq F_s K/30$ and if we recall that $K \geq 10$ we can conclude that we need to collect $N \geq F_s/3$ samples. The number of samples that we need to collect therefore grows with the sampling frequency, but the time it takes to collect these samples is $N/F_s = 1/3$ seconds for all sampling frequencies F_s.