MVE136 Random Signals Analysis

Written exam Wednesday 26 October 2016 2–6 pm

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AIDS: Beta or 2 sheets (=4 pages) of hand-written notes (computer print-outs and/or xerox-copies are not allowed), but not both these aids.

GRADES: 12 (40%), 18 (60%) and 24 (80%) points for grade 3, 4 and 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Task 1. Let $X(t)$ be a zero-mean and unit-variance stationary Gaussian random process with autocorrelation function $R_{XX}(\tau)$ and form a new random process $Y(t) = X(t)^2$. Express the cross-correlation function $R_{YX}(\tau) = E[Y(t)X(t+\tau)] = E[X(t)^2 X(t+\tau)]$ in terms of $R_{XX}(\tau)$. [Hint: Make use of the fact that $X(t)^2$ and $X(t+\tau) - R_{XX}(\tau)X(t)$ are independent, since $X(t)$ and $X(t+\tau) - R_{XX}(\tau)X(t)$ are independent since they are Gaussian and uncorrelated.] **(5 points)**

Task 2. Consider a Poisson process $X(t)$ with arrival rate $\lambda > 0$. Conditional on the information that there is exactly one arrival in the time interval $[0,1]$, show that the PDF of that arrival time T is uniformly distributed over the unit interval. [Hint: Note that $f_T(t) = \frac{d}{dt}\Pr(T \leq t) = \frac{d}{dt}\Pr(X(t)=1 \,|\, X(1)=1)$.] **(5 points)**

Task 3. A hot dog vendor operates a hot dog stand where the number of hot dogs he sells each day is modeled as a Poisson random variable with expected value α. Let $X[k]$ represent the number of hot dogs the vendor has in stock at the beginning of each day. At the end of the day, if his stock of hot dogs has fallen below some minimal value β, then the vendor immediately purchases enough new hot dogs to bring up his total stock to γ for the next day. On the other hand, if at the end of the day the stock of hot dogs is at least β, then the stock is not increased for the next days sails. Fine the transition matrix for the Markov chain $X[k]$. **(5 points)**

Task 4. Consider the function $R : \mathbb{Z} \to \mathbb{R}$ given by $R(0) = 1$, $R(\pm 1) = \alpha$ and $R(\pm k) = 0$ for $k = 2, 3, \ldots$. For which values of the real number α is $R(k)$ the autocorrelation function of some WSS random process? **(5 points)**

Task 5. The input to a continuous time LTI system is a WSS continuous time random process $X(t)$ with autocorrelation function

$$R_{XX}(\tau) = \frac{A\omega_0}{\pi} \frac{\sin(\omega_0 \tau)}{\omega_0 \tau} \quad \text{for } \tau \in \mathbb{R},$$

where $A, \omega_0 > 0$ are constants, whilst the LTI system has impulse response of the same form

$$h(t) = \frac{\omega_1}{\pi} \frac{\sin(\omega_1 t)}{\omega_1 t} \quad \text{for } t \in \mathbb{R},$$

where $\omega_1 > 0$ is a constant. Find the autocorrelation function of the output $Y(t)$ from the filter. **(5 points)**

Task 6. Suppose that $s[n]$ is an autoregressive (AR) process of second order with the parameters σ_s^2, a_1 and a_2, and that $w[n]$ is a moving average (MA) process of second order with the parameters σ_w^2, b_1 and b_2. Your task is to find the best possible Wiener filter that estimates $s[n]$ from $x[n] = s[n] + w[n]$, that is, $x[n]$ is the input to the Wiener filter and the output should be estimates of $s[n]$. You can assume that $s[n]$ and $w[n]$ are uncorrelated.

One could imagine using a causal, anti-causal or non-causal filter which either has a finite or infinite impulse response; which type of filter do you think has the potential of yielding the smallest errors? Please motivate. Also, write down at least one description of the optimal filter in terms of its difference equation, transfer function or frequency responce (one of these three descriptions is enough). **(5 points)**

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Solutions to written exam 26 October 2016

Task 1. We have $R_{YX}(\tau) = E[X(t)^2 X(t+\tau)] = E[X(t)^2(X(t+\tau) - R_{XX}(\tau)X(t))] + E[X(t)^2 R_{XX}(\tau)X(t))] = E[X(t)^2]\, E[X(t+\tau) - R_{XX}(\tau)X(t)] + R_{XX}(\tau)\, E[X(t)^3] = 0 + 0 = 0$.

Task 2. We have $f_T(t) = \frac{d}{dt}\Pr(T \le t) = \frac{d}{dt}\Pr(X(t)=1 \,|\, X(1)=1) = \frac{d}{dt}\Pr(X(t) = 1,\, X(1) = 1)/\Pr(X(1) = 1) = \frac{d}{dt}\Pr(X(t) = 1,\, X(1) - X(t) = 0)/\Pr(X(1) = 1) = \frac{d}{dt}\Pr(X(t) = 1)\Pr(X(1) - X(t) = 0)/\Pr(X(1) = 1) = \frac{d}{dt}\Pr(X(t) = 1)\Pr(X(1-t) = 0)/\Pr(X(1)=1) = \frac{d}{dt}(\lambda t)\,\mathrm{e}^{-\lambda t}\,\mathrm{e}^{-\lambda(1-t)}/(\lambda\,\mathrm{e}^{-\lambda}) = \frac{d}{dt}\, t = 1$ for $t \in [0,1]$.

Task 3. The states (possible values) for $X[k]$ are $\{\beta,\ldots,\gamma\}$ and lettning Y denote a Poisson random variable with expected value α we have $p_{i,j} = \Pr(i - Y = j) = \alpha^{i-j}\,\mathrm{e}^{-\alpha}/((i-j)!)$ for $j = \beta,\ldots,i$ and $p_{i,\gamma} = \Pr(i - Y < \beta) = \sum_{k=i-\beta+1}^{\infty} \alpha^k\,\mathrm{e}^{-\alpha}/(k!)$ while $p_{i,j} = 0$ for $j = i+1,\ldots,\gamma-1$ unless $i = j = \gamma$ in which case we instead have $p_{\gamma,\gamma} = \Pr(Y=0) + \Pr(Y > \gamma - \beta) = \mathrm{e}^{-\alpha} + \sum_{k=\gamma-\beta+1}^{\infty} \alpha^k\,\mathrm{e}^{-\alpha}/(k!)$.

Task 4. For $R(k)$ to be an autocorrelation function it is necessary and sufficient that its discrete time Fourier transform $S(f) = \sum_{k=-\infty}^{\infty} \mathrm{e}^{-j2\pi fk} R(k) = 1 + \alpha\,(\mathrm{e}^{-j2\pi f} + \mathrm{e}^{j2\pi f}) = 1 + 2\alpha\cos(2\pi f)$ in nonnegative, which in turn happens if and only if $|\alpha| \le 1/2$.

Task 5. We have $S_{XX}(f) = ((A\omega_0)/\pi)\,\mathrm{rect}(f/\omega_0)/\omega_0 = A\,\mathrm{rect}(f/\omega_0)/\pi$ and similarly $H(f) = \mathrm{rect}(f/\omega_1)/\pi$, so that $S_{YY}(f) = |H(f)|^2 S_{XX}(f) = A\,\mathrm{rect}(f/\omega_1)^2\,\mathrm{rect}(f/\omega_0)$ $/\pi^3 = A\,\mathrm{rect}(f/\min\{\omega_0,\omega_1\})/\pi^3$ which corresponds to

$$R_{YY}(\tau) = \frac{A\min\{\omega_0,\omega_1\}}{\pi^3}\,\frac{\sin(\min\{\omega_0,\omega_1\}\tau)}{\min\{\omega_0,\omega_1\}\tau} \quad \text{for } \tau \in \mathbb{R}.$$

Task 6. We can obtain the best possible performance by considering a non-causal, infinite impulse response (IIR) filter,

$$d[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k].$$

A short and simple argument for this is that the non-causal IIR filter has access to more measurements than the other filters and should therefore be able to produce better estimates. Also note that the other filter types, such as finite impulse response filters,

$$d_{\mathrm{FIR}}[n] = \sum_{k=0}^{N-1} h_{\mathrm{FIR}}[k]x[n-k],$$

are special cases of the non-causal IIR filter where we have introduced constraints on the filter coefficients, e.g.,

$$h_{\text{FIR}}[k] = 0 \quad \text{for } k \notin \{0, 1, 2, \ldots, N-1\}.$$

Clearly, introducing such constraints does not enable us to reach a better solution (a solution with smaller errors) to the optimisation problem

$$\min_{\{h[k]\}} E\left[(s[n] - d[n])^2\right].$$

For non-causal IIR Wiener filters, we know that the optimal filter has the frequency response

$$H\left(e^{j\omega}\right) = \frac{P_{dx}(e^{j\omega})}{P_x(e^{j\omega})}.$$

Since $d[n] = s[n]$, and $w[n]$ and $s[n]$ are uncorrelated, it holds that $P_{dx}(e^{j\omega}) = P_s(e^{j\omega})$, and since it also holds that $x[n] = s[n]+w[n]$, it follows that $P_x(e^{j\omega}) = P_s(e^{j\omega})+P_w(e^{j\omega})$.

The power spectral densities (PSD's) of $s[n]$ and $w[n]$ are easily obtained from the fact that they are AR and MA processes:

$$\begin{aligned} P_s(e^{j\omega}) &= \frac{\sigma_s^2}{\left|1 + a_1 e^{j\omega} + a_2 e^{j2\omega}\right|^2} \\ P_w(e^{j\omega}) &= \sigma_w^2 \left|1 + b_1 e^{j\omega} + b_2 e^{j2\omega}\right|^2. \end{aligned}$$

Putting the pieces together, we get the final expression for the frequency response of the optimal non-causal IIR Wiener filter

$$H\left(e^{j\omega}\right) = \frac{\sigma_s^2}{\sigma_s^2 + \sigma_w^2 \left|1 + a_1 e^{j\omega} + a_2 e^{j2\omega}\right|^2 \left|1 + b_1 e^{j\omega} + b_2 e^{j2\omega}\right|^2}.$$