MVE136 Random Signals Analysis

Written exam Tuesday 20 December 2016 2–6 pm

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AIDS: Beta or 2 sheets (=4 pages) of hand-written notes (computer print-outs and/or xerox-copies are not allowed), but not both these aids.

GRADES: 12 (40%), 18 (60%) and 24 (80%) points for grade 3, 4 and 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Task 1. The input to a discrete time LTI system is a WSS white noise process $X[k]$ with autocorrelation function $R_{XX}(k) = \delta(k)$, whilst the output is given by $Y[k] = \frac{1}{n}\sum_{m=0}^{n-1} X[k-m]$ for some interger $n \geq 1$. Show that the PSD of the output is $S_{YY}(f) = \frac{1}{n} + \frac{2}{n^2}\sum_{m=1}^{n-1}(n-m)\cos(2\pi m f)$. **(5 points)**

Task 2. Let $X(t)$ be a continuous time WSS zero-mean Gaussian random process and form a new process $Y(t) = X(t)\cos(\omega t + \Theta)$ where ω is a constant and Θ is a random variable that is uniformly distributed over $[0, 2\pi)$ and independent of the process $X(t)$. Is $Y(t)$ WSS? Is $Y(t)$ Gaussian? (Remember that answers must be motivated - a correct answer without a correct motivation gives no points!) [Hint: Recall that $2\cos(x)\cos(y) = \cos(x-y) + \cos(x+y)$.] **(5 points)**

Task 3. For a Markov chain $X[k]$, prove or disprove the following statement:

$$\Pr\big(X[k] = i_k \,\big|\, X[k+1] = i_{k+1}, \ldots, X[k+m] = i_{k+m}\big) = \Pr\big(X[k] = i_k \,\big|\, X[k+1] = i_{k+1}\big).$$

(5 points)

Task 4. Which of the following five functions $R_1(\tau) = \mathrm{e}^{-|\tau|}$, $R_2(\tau) = \mathrm{e}^{-\tau}u(\tau)$ (where $u(\tau) = 1$ for $\tau \geq 0$ and $u(\tau) = 0$ for $\tau < 0$), $R_3(\tau) = \mathrm{e}^{|\tau|}$, $R_4(\tau) = \cos(\tau)$ and $R_5(\tau) = \mathrm{sinc}(\tau) = \sin(\tau)/\tau$ could be the autocorrelation function of a WSS random process? (Remember that answers must be motivated - a correct answer without a correct motivation gives no points!) **(5 points)**

Task 5. Suppose that we observe a zero-mean WSS continuous time random process $X(t)$ with autocorrelation function $R_{XX}(\tau)$ over a time interval $(-\infty, t)$. Based on this observation we wish to predict a future process value $X(t+t_0)$ for a $t_0 > 0$ by means of a filter with impulse response $h(t)$ whose output $\int_0^\infty h(u)\, X(t-u)\, du$ will be our estimate of $X(t+t_0)$. Find and equation (which do not have to be solved) for the $h(u)$ that minimizes the mean-square error $\mathbf{E}\{(X(t+t_0) - \int_0^\infty h(u)\, X(t-u)\, du)^2\}$. **(5 points)**

Task 6. The topic in this problem is spectral estimation and periodogram averaging: Consider a wide sense stationary signal $x[n]$, with the power spectral density $P_x(e^{j\omega})$, illustrated in Figure 1 (a).

A friend of you, who does not have access to the true power spectral density, has collected 1024 samples of $x[n]$, based on which the periodogram has been computed. Your friend also decided to compute two averaged periodograms using Bartlett's method, where the data was splitted into $K = 16$ and $K = 64$, non-overlapping segments, respectively. In total your friend has thus computed three estimates of the power spectral density: the periodogram, and the averaged periodograms with $K = 16$ and $K = 64$. The three estimates are illustrated in Figure 1 (b)–(d), but we do not know which estimate is plotted in which figure. Please help your friend figure out the correspondence between the three estimates and these figures. Note that it is important to clearly motivate your answer in order to receive any credits. Finally, we would also like you to clarify the pros and cons with using a large versus a small value for K in terms of bias and variance of the estimates. **(5 points)**

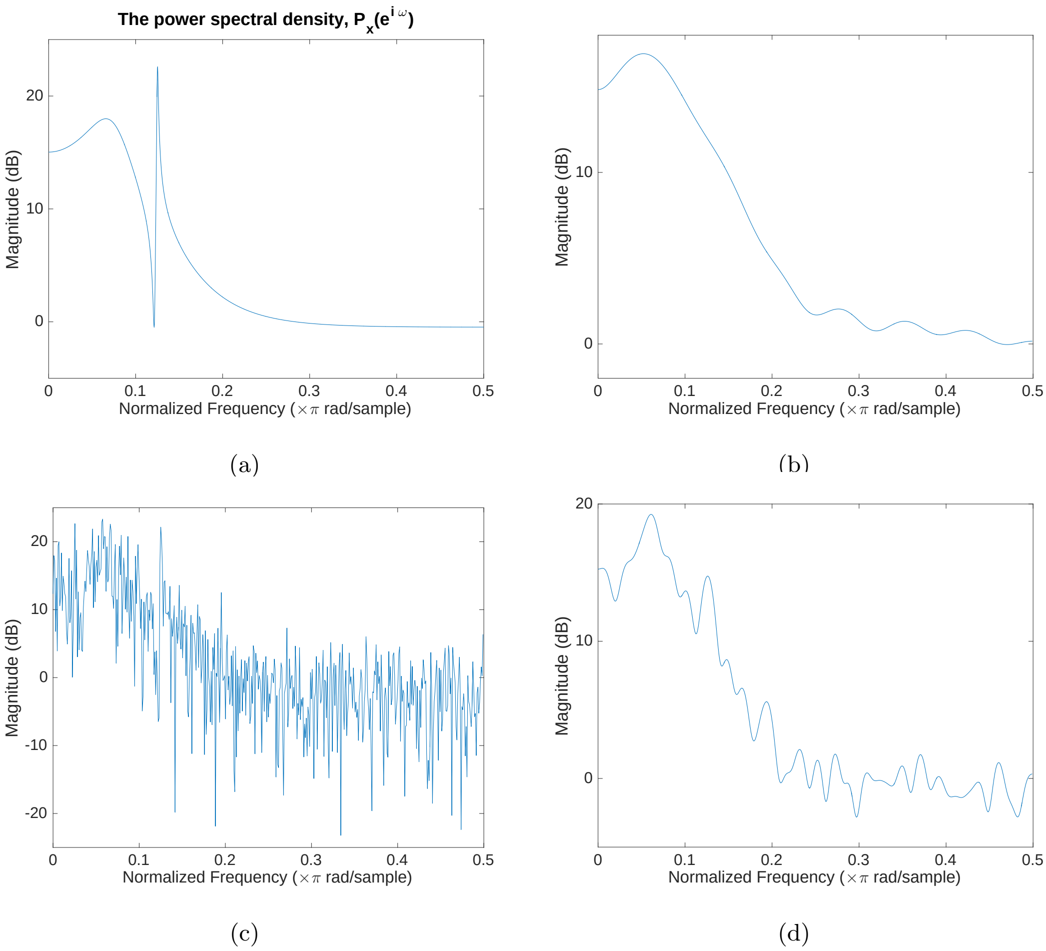

Figure 1: The true power spectral density, $P_x(e^{j\omega})$, is illustrated in (a), whereas the content in (b), (c) and (d) show different version of the periodogram.

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Solutions to written exam 20 December 2016

Task 1. It is easy to see that $R_{YY}(k) = (n-|k|)/n^2$ for $k = 0, \pm 1, \ldots, n-1$ while $R_{YY}(k) = 0$ for $|k| \geq n$, which in turn gives $S_{YY}(f) = \frac{1}{n^2}\sum_{m=-(n-1)}^{n-1}(n-|m|)\,\mathrm{e}^{-j2\pi fm}$ $= \frac{1}{n} + \frac{2}{n^2}\sum_{m=1}^{n-1}(n-m)\cos(2\pi mf)$.

Task 2. We have $\mathbf{E}\{Y(t)\} = \mathbf{E}\{X(t)\cos(\omega t+\Theta)\} = \mathbf{E}\{X(t)\}\,\mathbf{E}\{\cos(\omega t+\Theta)\} = 0$ and similarly $\mathbf{E}\{Y(s)Y(t)\} = \mathbf{E}\{X(s)X(t)\}\,\mathbf{E}\{\cos(\omega s+\Theta)\cos(\omega t+\Theta)\} = R_{XX}(t-s)$ $\mathbf{E}\big\{\frac{1}{2}\cos(\omega(t-s)) + \frac{1}{2}\cos(\omega(s+t)+2\Theta)\big\} = \frac{1}{2}\,R_{XX}(t-s)\,\cos(\omega(t-s))$ as $\mathbf{E}\{\cos(\alpha+2\Theta)\}$ $= 0$ by symmetry reasoning for any non-random constant α, so $Y(t)$ is WSS. However, $Y(t)$ is not Gaussian (unless $X(t) \equiv 0$) as $Y(0) = X(0)\cos(\Theta)$ and $Y(3\pi/(2\omega)) =$ $X(3\pi/(2\omega))\cos(3\pi/2+\Theta) = X(3\pi/(2\omega))\sin(\Theta)$ are uncorrelated $\mathbf{E}\{Y(0)Y(3\pi/(2\omega))\}$ $= \frac{1}{2}\,R_{XX}(3\pi/(2\omega))\cos(3\pi/2) = 0$ but clearly are not independent.

Task 3. The statement is true because

$$
\begin{aligned}
&\Pr\big(X[k]=i_k \,\big|\, X[k+1]=i_{k+1},\ldots,X[k+m]=i_{k+m}\big)\\
&\quad= \frac{\Pr\big(X[k]=i_k,\, X[k+1]=i_{k+1},\ldots,X[k+m]=i_{k+m}\big)}{\Pr\big(X[k+1]=i_{k+1},\ldots,X[k+m]=i_{k+m}\big)}\\
&\quad= \frac{\Big(\prod_{\ell=1}^{m}\Pr\big(X[k+\ell]=i_{k+\ell}\,\big|\,X[k+\ell-1]=i_{k+\ell-1}\big)\Big)\Pr\big(X[k]=i_k\big)}{\Big(\prod_{\ell=2}^{m}\Pr\big(X[k+\ell]=i_{k+\ell}\,\big|\,X[k+\ell-1]=i_{k+\ell-1}\big)\Big)\Pr\big(X[k+1]=i_{k+1}\big)}\\
&\quad= \frac{\Pr\big(X[k+1]=i_{k+1}\,\big|\,X[k]=i_k\big)\Pr\big(X[k]=i_k\big)}{\Pr\big(X[k+1]=i_{k+1}\big)}\\
&\quad= \frac{\Pr\big(X[k+1]=i_{k+1},\, X[k]=i_k\big)\Pr\big(X[k]=i_k\big)}{\Pr\big(X[k+1]=i_{k+1}\big)}\\
&\quad= \Pr\big(X[k]=i_k\,\big|\,X[k+1]=i_{k+1}\big).
\end{aligned}
$$

Task 4. The function R_2 does not fit with Property 8.4.2 in Miller and Childers evneness of autocorrelation functions while the function R_3 does not fit with Property 8.4.3 in Miller and Childers of autocorrelation functions (as it is concave). Hence neither of these two functions are autocorrelation functions. On the other hand the functions R_1, R_4 and R_5 all feautre repeatedly as examples of autocorrelation functions in the course and are thus autocorrelation functions all three of them. This can also be established directly by means of checking that their Fourier transform is non-negative real-valued and symmetric, which in turn follows from inspection of Table E.1 in Miller and Childers.

Task 5. According to the orthogonality principle the observed signal $X(s)$ will be orthogonal to the prediction error $X(t+t_0) - \int_0^\infty h(u)\, X(t-u)\, du$ for each $s \in (-\infty, t)$, which is to say that $R_{XX}(t+t_0-s) - \int_0^\infty h(u)\, R_{XX}(t-u-s)\, du = 0$ for $s \in (-\infty, t)$.

Task 6. The periodogram has two important weaknesses: 1) for finite data sequence, it has a bias in terms of finite resolution and 2) it has a large variance. The first weakness disappears as the length, N, of the collected data sequence goes to infinity, since the periodogram is asymptotically unbiased, but the variance converges to a constant, large value as N increases. The idea behind periodogram averaging is to split the sequence into many shorter sequences, and calculate the average of the periodograms for all of the shorter sequences. The periodograms of the shorter sequences typically have larger bias (since the sequences are shorter) but almost the same variance as the periodogram of the larger sequences. By taking the average of these periodograms, we can thus reduce the variance significantly but unfortunately we also increase the bias.

Among the three subfigures (a), (b) and (c), it is clear that (c) is by far the "noisiest", whereas (b) has a much smoother shape. In fact, the curve in subfigure (b) is so smooth that we have not even been able to resolve the two peaks in the true power spectral density. The estimate illustrated in (b) is thus Bartlett's method with $K = 64$, subfigure (c) shows the periodogram, whereas subfigure (d) contains the result from using Bartlett's method with $K = 16$. The trade-off that we are facing is that we obtain a smoother curve when we increase K, but that we lose resolution.