MVE136 Random Signals Analysis

Written exam Monday 14 August 2017 2–6 pm

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AIDS: Beta or 2 sheets (=4 pages) of hand-written notes (computer print-outs and/or xerox-copies are not allowed), but not both these aids.

GRADES: 12 (40%), 18 (60%) and 24 (80%) points for grade 3, 4 and 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Task 1. Let $X(t)$ and $Y(t)$ be independent Poisson processes with unit arrival rate. Calculate the probability $P[X(1) = 1 \,|\, X(2)+Y(2) = 4]$. [**Hint:** Remember that if U and V are independent Poisson distributed random variables with means μ_U and μ_V, respectively, then $U+V$ is Poisson distributed with mean $\mu_U+\mu_V$.] **(5 points)**

Task 2. A Markov chain has two states 0 and 1 and transition matrix

$$P = \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix}.$$

The initial value of the chain is 0 with probability 2/3 and 1 with probability 1/3. What is the probability that the chain after two steps is in state 1? **(5 points)**

Task 3. A WSS continuous time random process $X(t)$ has PSD $S_{XX}(f) = \mathrm{e}^{-|f|}$. What is the PSD of the derivative process $X'(t)$? **(5 points)**

Task 4. A WSS continuous time Gaussian random process $X(t)$ has autocorrelation function $R_{XX}(\tau) = \mathrm{e}^{-|\tau|}$ and in addition it holds that $P[X(1)+X(2) \leq 3] = 4/5$. Find $E[X(t)]$. **(5 points)**

Task 5. A WSS continuous time random process $X(t)$ with autocorrelation function $R_{XX}(\tau) = \mathrm{e}^{-|\tau|}$ has been observed at times 0 and 2 and the task is to use these observations to form a linear estimator $\overline{X(1)} = a\,X(0) + b\,X(2)$ of $X(1)$ that minimizes the mean-square error $E\big[(\overline{X(1)} - X(1))^2\big]$ where a and b are real numbers. Find the values of a and b. **(5 points)**

Task 6. Explain what weakness of the periodogram that the modified periodogram tries to remove. What is the difference (/are the differences) between the periodogram and the modified periodogram? **(5 points)**

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Solutions to written exam 14 August 2017

Task 1. $P[X(1)=1\,|\,X(2)+Y(2)=4] = P[X(1)=1,\ X(2)+Y(2)=4]/P[X(2)+Y(2)=4] = P[X(1)=1,\ (X(2)-X(1))+X(1)+Y(2)=4]/P[\mathrm{Po}(4)=4] = P[\mathrm{Po}(1)=1]\,P[\mathrm{Po}(3)=3]/P[\mathrm{Po}(4)=4] = (\mathrm{e}^{-1}\,1^1/(1!))\,(\mathrm{e}^{-3}\,3^3/(3!))/(\mathrm{e}^{-4}\,4^4/(4!)) = 27/64$.

Task 2. We have $\pi(2)_1 = (\pi(0)\,P^2)_1 = \ldots = 31/48$.

Task 3. $S_{X'X'}(f) = F[R_{X'X'}](f) = F[-R''_{XX}](f) = -(j2\pi f)^2 S_{XX}(f) = (2\pi f)^2 S_{XX}(f)$.

Task 4. Writing $\mu = E[X(t)]$ we have $E[X(1)+X(2)] = 2\mu$ and $Var[X(1)+X(2)] = 2\,R_{XX}(0)+2\,R_{XX}(1) = 2\,(1+\mathrm{e}^{-1})$ so that $P[X(1)+X(2)\leq 3] = \Phi\big((3-2\mu)/(2\,(1+\mathrm{e}^{-1}))\big)$ giving $(3-2\mu)/(2\,(1+\mathrm{e}^{-1})) = \Phi^{-1}(4/5)$ and $\mu = \big(3-(2\,(1+\mathrm{e}^{-1}))\,\Phi^{-1}(4/5)\big)/2$.

Task 5. By symmetry $a=b$ so that $E\big[(\overline{X(1)}-X(1))^2\big] = 1+2\,a^2-4\,a\,\mathrm{e}^{-1}+2\,a^2\,\mathrm{e}^{-2}$ giving $a=b=\mathrm{e}^{-1}/(1+\mathrm{e}^{-2})$.

Task 6. The periodogram has limited resolution whenever the sampled (observed) sequence has finite length, N. Perhaps the most serious problem is that weak frequency components can be hidden/masked in the sidelobes of a strong frequency component. The modified periodogram tries to limit this "frequency masking". Another way to put it, is that the periodogram has a bias for finite N and that the modified periodogram tries to reduce the bias.

The difference between the periodogram and the modified periodogram can be explained as follows. The original periodogram is the DTFT

$$\hat{P}_{per}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} w_R[n]x[n]\,\mathrm{e}^{j\omega n},$$

where

$$w_R[n] = \begin{cases} 1 & \text{for } 0\leq n\leq N-1 \\ 0 & \text{otherwise.} \end{cases}$$

This rectangular window is introduced as one way to describe that we have only observed $x[n]$ for $n=0,1,\ldots,N-1$. One problem with $w_R[n]$ is that its Fourier transform, $W_R(e^{j\omega})$, has very large sidelobes which give rise to the frequency masking mentioned above. The modified periodogram reduces the sidelobes by replacing $w_R[n]$ with a different window function that has lower sidelobes (but unfortunately the alternative windows always have wider mainlobes).