MVE136 Random Signals Analysis

Written exam Tuesday 19 December 2017 2–6 pm

TEACHER: Patrik Albin. JOUR: Fanny Berglund, telephone 031 7725325.

AIDS: Beta or 2 sheets (=4 pages) of hand-written notes (computer print-outs and/or xerox-copies are not allowed), but not both these aids.

GRADES: 12 (40%), 18 (60%) and 24 (80%) points for grade 3, 4 and 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Task 1. Find the autocorrelation function $R_{XX}(m,n)$ of the discrete time random process $X[n]$ given by $X[0]=0$ and $X[n]=\sum_{k=1}^{n} W_k$ for $n=1,2,\ldots$, where $W_1,W_2,W_3,\ldots$ are uncorrelated zero-mean unit-variance random variables. **(5 points)**

Task 2. Calculate the mean function $\mu_X(t)$ and the autocorrelation function $R_{XX}(s,t)$ for the continuous time random process $X(t)$, $t\geq 0$, given by $X(t)=\mathrm{e}^{-At}$, where A is a unit mean exponentially distributed random variable. **(5 points)**

Task 3. A Wiener process is a zero-mean continuous time Gaussian process $W(t)$, $t\geq 0$, with autocorrelation function $R_{WW}(s,t)=\min(s,t)$. Prove that the Wiener process has independent increments, which is to say, that $W(t_4)-W(t_3)$ and $W(t_2)-W(t_1)$ are independent random variables for $0\leq t_1\leq t_2\leq t_3\leq t_4$. **(5 points)**

Task 4. For a Markov chain with each of the transition probability matrices (a)-(c) given below, find the states that communicate with each other together with the periodicities of these various states.

$$\text{(a)}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 1/3 & 1/3 & 1/3 & 0 \end{bmatrix} \quad \text{(b)}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1/3 & 0 & 2/3 & 0 \end{bmatrix} \quad \text{(c)}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1 \\ 0 & 1/2 & 1/2 & 0 \end{bmatrix}$$

(5 points)

Task 5. Let a continuous time white noise process $N(t)$, $t\in\mathbb{R}$, be input to an LTI system with impulse response $h(t)=2/(1+(2\pi t)^2)$ for $t\in\mathbb{R}$. Find the average power $E[Y(t)^2]$ of the output $Y(t)$, $t\in\mathbb{R}$, from the LTI system. **(5 points)**

Task 6. Compute the periodogram of the data sequence $x[0]=2$, $x[1]=0$ and $x[2]=2$.

What property of the periodogram would improve if the number of samples N were increased from $N=3$ and what weaknesses would remain? **(5 points)**

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Solutions to written exam 19 December 2017

Task 1. According to Example 8.14 in the book by Miller and Childers we have $R_{XX}(m,n) = \min(m,n)$.

Task 2. $\mu_X(t) = E[X(t)] = E[\mathrm{e}^{-At}] = \int_0^\infty \mathrm{e}^{-at}\,\mathrm{e}^{-a}\,da = 1/(1{+}t)$ and $R_{XX}(s,t) = E[X(s)X(t)] = E[\mathrm{e}^{-As}\mathrm{e}^{-At}] = \ldots = 1/(1{+}s{+}t)$.

Task 3. As $W(t_4){-}W(t_3)$ and $W(t_2){-}W(t_1)$ are (jointly) Gaussian random variables they are independent if they are uncorrelated. That their correlation is zero, in turn, follows from noting that $E[(W(t_4){-}W(t_3))\,(W(t_2){-}W(t_1))] = C_{WW}(t_4,t_2){-}C_{WW}(t_3,t_2){-}C_{WW}(t_4,t_1){+}C_{WW}(t_3,t_1) = R_{WW}(t_4,t_2){-}R_{WW}(t_3,t_2){-}R_{WW}(t_4,t_1){+}R_{WW}(t_3,t_1) = \min(t_4,t_2) - \min(t_3,t_2) - \min(t_4,t_1) + \min(t_2,t_1) = t_2 - t_2 - t_1 + t_1 = 0$.

Task 4. By inspection, for chain (a) the first three states communicate with common period 1, for chain (b) all four states communicate with common period 3, while for chain (c) all four states communicate with common period 2.

Task 5. As $H(f) = (Fh)(f) = \mathrm{e}^{-|f|}$ we have $E[Y(t)^2] = \int_{-\infty}^\infty S_{YY}(f)\,df = \int_{-\infty}^\infty |H(f)|^2\, S_{NN}(f)\,df = (N_0/2)\int_{-\infty}^\infty \mathrm{e}^{-2|f|}\,df = N_0/2$.

Task 6. The periodogram is given by $\hat{P}_{per}(e^{j\omega}) = \frac{1}{N}\,|X_N(e^{j\omega})|^2$, where, in this case, $N=3$ and

$$X_N(e^{j\omega}) = X_3(e^{j\omega}) = x[0] + x[1]\,e^{-j\omega} + x[2]\,e^{-2j\omega} = 2\,(1 + e^{-2j\omega}).$$

It follows that

$$\hat{P}_{per}(e^{j\omega}) = \frac{4}{3}\,(1 + e^{-2j\omega})\,(1 + e^{2j\omega}) = \frac{4}{3}\,(2 + e^{-2j\omega} + e^{2j\omega}) = \frac{8}{3}\,(1 + \cos(2\omega)).$$

The periodogram is asymptotically unbiased and therefore the periodogram improves as N grows in the sense that the bias goes to zero. However, the variance of the periodogram converges to a fixed non-zero value as N grows and the remaining weakness of the periodogram is hence the persisting variance of the estimates.