MVE136 Random Signals Analysis

Written exam Sunday 26 January 2014 1 – 5 pm

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Aids: Beta or 2 sheets (4 pages) of hand-written notes (computer print-outs and/or xerox-copies are not allowed), but not both these aids.

Grades: 12 (40%), 18 (60%) and 24 (80%) points for grade 3, 4 and 5, respectively.

Good luck!

Task 1. Let A be a random variable with finite mean and finite variance that is independent of another random variable Θ that is uniformly distributed over the interval $[0, 2\pi]$. Consider the discrete time random process $\{X_n, n \in \mathbb{Z}\}$ given by $X_n = A\cos(\Theta n)$. Show that the process X_n is not wide-sense stationary (WSS). (Hint: The formula $\cos(x)\cos(y) = \frac{1}{2}\cos(x+y) + \frac{1}{2}\cos(y-x)$ can be useful.) **(5 points)**

Task 2. A Wiener process (also called Brownian motion) is a zero-mean continuous time Gaussian process $\{W(t), t \geq 0\}$ with autocorrelation function $R_{WW}(s,t) = \min(s,t)$. [Hence the autocorrelation agrees with the autocovariance function of a Poisson processes with arrival rate (/intensity) 1.] Prove that the Wiener process has independent increments, which is to say, that $W(t_4) - W(t_3)$ and $W(t_2) - W(t_1)$ are independent random variables for $0 \leq t_1 \leq t_2 \leq t_3 \leq t_4$. **(5 points)**

Task 3. Find the probability $P(X(t) + X(2t) = 5)$ when $\{X(t), t \geq 0\}$ is a Poisson process with arrival rate (/intensity) $\lambda > 0$. **(5 points)**

Task 4. Consider a so called discrete time random walk process $\{X[n], n \geq 0\}$ on the (both positive and negative) integers that starts at the origin (that is, $X[0] = 0$). At each time instant, the process either increases by 1 with probability p or decreases by 1 with probability $1-p$. Explain why this process is a Markov chain and find its transition probabilities $p_{i,j}$. Are the states of this chain periodic or aperiodic? If they are periocid, what is the period? **(5 points)**

Task 5. A wide sense stationary (WSS) continuous time signal process $Z(t)$ with power spectral density (PSD) $S_{ZZ}(f)$ is sent on a noisy channel with an additive WSS noise process $N(t)$ that is independent of $Z(t)$ and has PSD $S_{NN}(f)$. Hence the recived noise disturbed signal process $X(t)$ is given by $X(t) = Z(t) + N(t)$. We send $X(t)$ through a so called Wiener filter with output process $Y(t)$ that is designed to minimize the mean square error $E[(Z(t) - Y(t))^2]$. What is the PSD of $Y(t)$? **(5 points)**

Task 6. Explain what weakness of the periodogram that the modified periodogram tries to remove. What is the difference (/are the differences) between the periodogram and the modified periodogram? **(5 points)**

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Solutions to written exam Sunday 26 January 2014

Task 1. We have $E(X_m X_n) = \frac{1}{2}E(A^2)\,E(\cos(\Theta(m+n)) + \frac{1}{2}E(A^2)\,E(\cos(\Theta(m-n)) = \frac{1}{2}E(A^2)\,\delta(m+n) + \frac{1}{2}E(A^2)\,\delta(m-n)$, which is not a functiuon of $m-n$ only, so that X_n is not WSS.

Task 2. As $W(t_4)-W(t_3)$ and $W(t_2)-W(t_1)$ are (jointly) Gaussian random variables they are independent if they are uncorrelated. That their correlation is zero, in turn, follows from noting that $E[(W(t_4)-W(t_3))\,(W(t_2)-W(t_1))] = C_{WW}(t_4,t_2)-C_{WW}(t_3,t_2)-C_{WW}(t_4,t_1)+C_{WW}(t_3,t_1) = R_{WW}(t_4,t_2)-R_{WW}(t_3,t_2)-R_{WW}(t_4,t_1)+R_{WW}(t_3,t_1) = \min(t_4,t_2)-\min(t_3,t_2)-\min(t_4,t_1)+\min(t_2,t_1) = t_2-t_2-t_1+t_1 = 0$.

Task 3. As $X(t)$ and $X(2t)-X(t)$ are independent Poisson distributed with parameter λt it follows that $P(X(t)+X(2t)=5) = P(X(2t)-X(t)=5)\,P(X(t)=0) + P(X(2t)-X(t)=3)\,P(X(t)=1) + P(X(2t)-X(t)=1)\,P(X(t)=2) = \frac{(\lambda t)^5}{5!}\,\mathrm{e}^{-\lambda t}\,\frac{(\lambda t)^0}{0!}\,\mathrm{e}^{-\lambda t} + \frac{(\lambda t)^3}{3!}\,\mathrm{e}^{-\lambda t}\,\frac{(\lambda t)^1}{1!}\,\mathrm{e}^{-\lambda t} + \frac{(\lambda t)^1}{1!}\,\mathrm{e}^{-\lambda t}\,\frac{(\lambda t)^2}{2!}\,\mathrm{e}^{-\lambda t} = (\frac{(\lambda t)^5}{120} + \frac{(\lambda t)^4}{6} + \frac{(\lambda t)^3}{2})\,\mathrm{e}^{-2\lambda t}$.

Task 4. This problem is solved in Example 9.11 in the book by Miller and Childers.

Task 5. As the Wiener filter has transfer function $H(f) = S_{ZZ}(f)/(S_{ZZ}(f)+S_{NN}(f))$ we have $S_{YY}(f) = |H(f)|^2 S_{XX}(f) = H(f)^2(S_{ZZ}(f)+S_{NN}(f)) = S_{ZZ}(f)^2/(S_{ZZ}(f)+S_{NN}(f))$.

Task 6. The periodogram has limited resolution whenever the sampled (observed) sequence has finite length, N. Perhaps the most serious problem is that weak frequency components can be hidden/masked in the sidelobes of a strong frequency component. The modified periodogram tries to limit this "frequency masking". Another way to put it, is that the periodogram has a bias for finite N and that the modified periodogram tries to reduce the bias.

The difference between the periodogram and the modified periodogram can be explained as follows. The original periodogram is the DTFT

$$\hat{P}_{per}(e^{j\omega}) = \sum_{n=-\infty}^{\infty} w_R[n]x[n]\,\mathrm{e}^{j\omega n},$$

where

$$w_R[n] = \begin{cases} 1 & \text{for } 0 \le n \le N-1 \\ 0 & \text{otherwise.} \end{cases}$$

This rectangular window is introduced as one way to describe that we have only observed $x[n]$ for $n = 0, 1, \ldots, N-1$. One problem with $w_R[n]$ is that its Fourier transform, $W_R(e^{j\omega})$, has very large sidelobes which give rise to the frequency masking mentioned above. The modified periodogram reduces the sidelobes by replacing $w_R[n]$ with a different window function that has lower sidelobes (but unfortunately the alternative windows always have wider mainlobes).