MVE136 Random Signals Analysis

Written exam Monday 5 January 2015 2 − 6 pm

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AIDS: Beta or 2 sheets (4 pages) of hand-written notes (computer print-outs and/or xerox-copies are not allowed), but not both these aids.

GRADES: 12 (40%), 18 (60%) and 24 (80%) points for grade 3, 4 and 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Task 1. Consider the discrete time random process given by $X[0] = 0$ and $X[n] = \sum_{k=1}^{n} W_k$ for $n = 1, 2, \ldots$, where $W_1, W_2, W_3, \ldots$ are independent identically distributed zero-mean unit-variance random variables. Find the ACF $R_{XX}(m, n)$. **(5 points)**

Task 2. Consider the continuous time random process $X(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$ for $t \in \mathbb{R}$, where A and B are independent zero-mean unit-variance Gaussian random variables and $\omega_0 \in \mathbb{R}$ is a constant. Find the joint PDF $f_{X(s),X(t)}(x, y)$ for the process values $(X(s), X(t))$. [Hint: $\cos(x)\cos(y) + \sin(x)\sin(y) = \cos(x-y)$.] **(5 points)**

Task 3. Prove that $\pi(k) = \pi(0)\, P^k$ for a Markov chain $X[k]$ with transition probability matrix P and distribution at time k given by the row matrix $\pi(k)$. **(5 points)**

Task 4. Consider the continuous time random process $X(t) = A\sin(\omega_0 t + \Theta)$ for $t \in \mathbb{R}$, where A and Θ are independent random variables with Θ uniformly distributed over the interval $[0, 2\pi]$ and $\omega_0 \in \mathbb{R}$ is a constant. Find the PSD $S_{XX}(f)$ of $X(t)$. [Hint: $\sin(x)\sin(y) = \frac{1}{2}\cos(x-y) - \frac{1}{2}\cos(x+y)$.] **(5 points)**

Task 5. A zero-mean WSS continuous time process $Z(t)$ with PSD $S_{ZZ}(f)$ is sent on a noisy channel were the noise $N(t)$ is independent of $Z(t)$ and zero-mean WSS with PSD $S_{NN}(f)$. The recived signal $X(t) = Z(t) + N(t)$ is input to a Wiener filter with output $Y(t)$. Express $E[(Z(t)-Y(t))^2]$ in terms of $S_{ZZ}(f)$ and $S_{NN}(f)$. **(5 points)**

Task 6. Consider FIR Wiener filtering when the desired signal is generated as $d[n] = e[n] - 0.5\, e[n-1]$, where $e[n]$ is a zero-mean white noise (wide sense stationary) process with variance $\sigma_e^2 = 1$. We observe $x[n] = d[n] + v[n]$, where $v[n]$ is zero-mean white noise with variance $\sigma_v^2 = 0.5$ which is uncorrelated with $e[n]$. Your task is to find the optimal filtering coefficients h_0 and h_1 which are such that $\hat{d}[n] = h_0\, x[n] + h_1\, x[n-1]$ minimizes the mean-square distance $E\big[(d[n] - \hat{d}[n])^2\big]$. **(5 points)**

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Solutions to written exam 5 January 2015

Task 1. According to Example 8.14 in the book by Miller and Childers we have $R_{XX}(m,n) = \min(m,n)$.

Task 2. This is the special case of Example 8.23 in the book by Miller and Childers with $\sigma^2 = 1$.

Task 3. This is Equation 9.12 in the book by Miller and Childers.

Task 4. According to Example 10.2 in the book by Miller and Childers we have $S_{XX}(f) = \frac{1}{4}\, E[A^2]\,(\delta(f-f_0)+\delta(f+f_0))$, where $f_0 = \omega_0/(2\pi)$.

Task 5. By basic calculations we have $E[(Z(t)-Y(t))^2] = \int_{-\infty}^{\infty}[|H(f)|^2(S_{ZZ}(f)+S_{NN}(f)) - 2H(f)S_{ZZ}(f) + S_{ZZ}(f)]\,df$. Upon insertion of the Wiener filter transfer function $H(f) = S_{ZZ}(f)/(S_{ZZ}(f)+S_{NN}(f))$ basic algebraic manupulations give $E[(Z(t)-Y(t))^2] = \int_{-\infty}^{\infty} S_{ZZ}(f)S_{NN}(f)/(S_{ZZ}(f)+S_{NN}(f))\,df$.

Task 6. Let us first rederive the Wiener-Hopf equations. According to the orthogonality principle, the optimal filter should satisfy

$$E\big[(d[n]-\hat{d}[n])\,x[n]\big] = 0 \qquad \text{and} \qquad E\big[(d[n]-\hat{d}[n])\,x[n-1]\big] = 0.$$

By incorporating the expression $\hat{d}[n] = h_0\,x[n] + h_1\,x[n-1]$ we obtain

$$h_0\,r_x[0] + h_1\,r_x[1] = r_{dx}[0] \quad \text{and} \quad h_0\,r_x[1] + h_1\,r_x[0] = r_{dx}[1].$$

To find the optimal filtering coefficients we therefore need to find $r_{dx}[0]$, $r_{dx}[1]$, $r_x[0]$ and $r_x[1]$: Considering that $v[n]$ and $e[n]$ are uncorrelated processeses it holds that $v[n]$ and $d[n]$ are uncorrelated, so that

$$r_{dx}[k] = E\big[d[n]\,x[n-k]\big] = E\big[d[n]\,(d[n-k]+v[n-k])\big] = r_d[k]$$
$$r_x[k] = E\big[(d[n]+v[n])\,(d[n-k]+v[n-k])\big] = r_d[k] + r_v[k]$$

for $k = 0,1$. As we know that $r_v[0] = \sigma_v^2 = 0.5$ and $r_v[1] = 0$ it only remains to compute $r_d[0]$ and $r_d[1]$. We can compute these directly from the fact that

$$r_d[k] = E\big[d[n]\,d[n-k]\big] = E\big[(e[n]+0.5\,e[n-1])\,(e[n-k]+0.5\,e[n-k-1])\big]$$

for $k = 0,1$, which yields $r_d[0] = 1.25\,\sigma_e^2 = 1.25$ and $r_d[1] = -0.5\,\sigma_e^2 = 0.5$. Hence we have

$$h_0\,1.75 + h_1\,(-0.5) = 1.25 \quad \text{and} \quad h_0\,(-0.5) + h_1\,1.75 = (-0.5),$$

from which we can find the solutions $h_0 = \frac{31}{45} \approx 0.689$ and $h_1 = -\frac{4}{45} \approx -0.089$.