MVE136 Random Signals Analysis

Written exam Friday 27 October 2017 8.30–12.30

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AIDS: Beta or 2 sheets (=4 pages) of hand-written notes (computer print-outs and/or xerox-copies are not allowed), but not both these aids.

GRADES: 12 (40%), 18 (60%) and 24 (80%) points for grade 3, 4 and 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Task 1. Calculate the mean function $\mu_X(t)$ and the autocorrelation function $R_{XX}(s,t)$ for the continuous time random process $X(t)$, $t \geq 0$, given by $X(t) = \mathrm{e}^{-Ut}$, where U is a random variable that is uniformly distributed over the interval $[0,1]$. **(5 points)**

Task 2. Is it possible for a two state Markov chain not to have a stationary distribution? **(5 points)**

Task 3. Let $W(t)$, $t \geq 0$, be a zero-mean continuous time Gaussian process with autocorrelation function $R_{WW}(s,t) = \min(s,t)$. Find the probability $\Pr\big(\int_0^1 W(t)\,dt > 1\big)$. **(5 points)**

Task 4. For a continuous time LTI system with WSS input and output $X(t)$ and $Y(t)$, respectively, with PSD's $S_{XX}(f)$ and $S_{YY}(f)$, respectively, and with transfer function $H(f)$ it holds that $S_{YY}(f) = |H(f)|^2 S_{XX}(f)$. Prove this fact. **(5 points)**

Task 5. A continuous time communication system transmits a square pulse $s(t) = 1$ for $t \in [0,1]$ and $s(t) = 0$ for $t \notin [0,1]$ on a noisy channel so that the recived signal is $R(t) = s(t) + N(t)$ where $N(t)$ is white noise. The recived signal $R(t)$ is processed through a matched filter with output $Y(t)$ that produces the optimum SNR at time $t = t_0$. Find the impulse response $h(t)$ of the matched filter as well as the value of $Y(t_0)$. **(5 points)**

Task 6. We have collected four measurements

$$x[0] = 0.3, \quad x[1] = -0.2, \quad x[2] = 0.1 \quad \text{and} \quad x[3] = -0.3,$$

of a discrete time WSS random process $\{x[n]\}_{n=-\infty}^{\infty}$ and we wish to find a mathematical model for the measured process. We select to use a simple AR(1)-model $x[n] + a_1\, x[n-1] = e[n]$ for $n \in \mathbb{Z}$, where $\{e[n]\}_{n=-\infty}^{\infty}$ is discrete time white noise: Your task is to estimate the parameter a_1 and the white noise variance $\sigma_e^2 = \mathbf{E}\{e[n]^2\}$. **(5 points)**

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Solutions to written exam 27 October 2017

Task 1. $\mu_X(t) = E[X(t)] = E[\mathrm{e}^{-Ut}] = \int_0^1 \mathrm{e}^{-ut}\,du = \ldots = (1-\mathrm{e}^{-t})/t$ for $t>0$ while (by inspection) $\mu_X(0) = 1$ and similarly $R_{XX}(s,t) = E[X(s)X(t)] = E[\mathrm{e}^{-Us}\mathrm{e}^{-Ut}] = \ldots = (1-\mathrm{e}^{-(s+t)})/(s+t)$ for $s+t>0$ with $R_{XX}(0,0) = 1$.

Task 2. A stationary distribution π exists if and only if we can solve

$$\begin{cases} [\pi_1\ \ \pi_2] = [\pi_1\ \ \pi_2]\begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix} \\ \pi_1+\pi_2 = 1 \end{cases} \Leftrightarrow \begin{cases} p\pi_1 = q\pi_2 \\ \pi_1+\pi_2 = 1 \end{cases} \Leftrightarrow \begin{cases} \begin{cases} \pi_1 = q/(p+q) \\ \pi_2 = p/(p+q) \end{cases} & \text{if } p+q>0 \\ \pi_1+\pi_2 = 1 & \text{if } p+q=0 \end{cases}.$$

As this always have at least one solution a stationary distribution always exists.

Task 3. As the random variable $\int_0^1 W(t)\,dt$ is Gaussian (being a linear combination of Gaussian process values) with mean $E[\int_0^1 W(t)\,dt] = \int_0^1 E[W(t)]\,dt = 0$ and variance $\mathrm{Var}(\int_0^1 W(t)\,dt) = \mathrm{Cov}(\int_0^1 W(s)\,ds, \int_0^1 W(t)\,dt) = \int_0^1\int_0^1 \mathrm{Cov}(W(s),W(t))\,dsdt = \int_0^1\int_0^1 \min(s,t)\,dsdt = \ldots = 1/3$ we have $\Pr\big(\int_0^1 W(t)\,dt > 1\big) = \Pr(\mathrm{N}(0,1/3) > 1) = 1-\Phi(\sqrt{3}\,)$.

Task 4. This is done in Section 11.1 of the book by Miller and Childers.

Task 5. According to Example 11.6 in the book by Miller and Childers we have $h(t) = s(t_0-t)$ and $Y(t_0) = (h\star R)(t_0) = 1 + \int_0^1 N(u)\,du$.

Task 6. As the autocorrelation function $r_x[k]$ of the AR(1)-process satisfies the Yule-Walker equations

$$a_1\,r_x[0] + r_x[1] = 0 \quad \text{and} \quad r_x[0] + a_1\,r_x[1] = \sigma_e^2,$$

we obtain estimates $\hat{a}_1$ and $\hat{\sigma}_e^2$ of the parameters a_1 and σ_e^2 by solving the equations

$$\hat{a}_1\,\hat{r}_x[0] + \hat{r}_x[1] = 0 \quad \text{and} \quad \hat{r}_x[0] + \hat{a}_1\,\hat{r}_x[1] = \hat{\sigma}_e^2$$

giving

$$\hat{a}_1 = -\hat{r}_x[1]/\hat{r}_x[0] \quad \text{and} \quad \hat{\sigma}_e^2 = \hat{r}_x[0] - \hat{r}_x[1]^2/\hat{r}_x[0],$$

where

$$\hat{r}_x[0] = \frac{1}{4}\sum_{n=0}^{3} x[n]^2 = \ldots = 0.0575 \quad \text{and} \quad \hat{r}_x[1] = \frac{1}{4}\sum_{n=0}^{2} x[n]\,x[n+1] = \ldots = -0.0275.$$