MVE136 Random Signals Analysis

Written exam Monday 20 August 2018 2-6 PM

Teacher: Patrik Albin. Jour: Olof Giselsson 031 7725325.

Aids: Beta <u>or</u> 2 sheets (=4 pages) of hand-written notes (computer print-outs and/or xerox-copies are not allowed), but not both these aids.

Grades: 12 (40%), 18 (60%) and 24 (80%) points for grade 3, 4 and 5, respectively.

Motivations: All answers/solutions must be motivated. Good luck!

Task 1. Calculate $P[X(2)=2 \mid X(1)=1,\, X(3)=3]$ for a unit rate/intensity Poisson process. **(5 points)**

Task 2. Calculate $P[X(1)+X(2) \geq 2]$ for a zero-mean WSS continuous time Gaussian random process $\{X(t)\}_{t\in\mathbb{R}}$ with autocorrelationfunction $R_{XX}(\tau) = \mathrm{e}^{-|\tau|}$. **(5 points)**

Task 3. A Markov chain $\{X(n)\}_{n=0}^{\infty}$ has two states 0 and 1 and transition matrix

$$P = \begin{bmatrix} p_{00} & p_{01} \\ p_{10} & p_{11} \end{bmatrix},$$

where $p_{00}, p_{01}, p_{10}, p_{11} \in (0,1)$. Find $P[X(1)=1 \mid X(0)=0,\, X(2)=0]$. **(5 points)**

Task 4. Let $S_{XX}(\omega)$ be the PSD of a WSS random process $X(t)$. Prove that $S_{XX}(\omega)$ is symmetric $S_{XX}(-\omega) = S_{XX}(\omega)$. **(5 points)**

Task 5. Derive the frequency responce $H(\omega)$ of the mathematical operation of differentiating a continuous time signal $\{x(t)\}_{t\in\mathbb{R}}$ with $\int_{-\infty}^{\infty} |x(t)|\, dt < \infty$. [In other words $H(\omega) = \int_{-\infty}^{\infty} \mathrm{e}^{-j2\pi\omega t} h(t)\, dt$, where $x'(t) = \int_{-\infty}^{\infty} h(t-s)x(s)\, ds$.] Note that it is a derivation that is asked for - it is not enough to just give a formula for $H(\omega)$. **(5 points)**

Task 6. Let $s[n]$ be a second order autoregressive (AR) process with parameters σ_s^2, a_1 and a_2 that is independent of a second order moving average (MA) process $w[n]$ with parameters σ_w^2, b_1 and b_2. Find the best possible Wiener filter that outputs estimates $y[n]$ of $s[n]$ from the input $x[n] = s[n] + w[n]$. **(5 points)**

MVE136 Random Signals Analysis

Solutions to written exam 20 August 2018

Task 1. $P[X(2)=2 \,|\, X(1)=1,\ X(3)=3] = P[X(1)=1,\ X(2)=2,\ X(3)=3]/P[X(1)=1,\ X(3)=3] = P[X(1)=1,\ X(2)-X(1)=1,\ X(3)-X(2)=1]/P[X(1)=1,\ X(3)-X(1)=2] = (\mathrm{e}^{-1})^3/(\mathrm{e}^{-1}\,\frac{2^2}{2!}\,\mathrm{e}^{-2}) = \frac{1}{2}$.

Task 2. We have $P[X(1)+X(2) \geq 2] = P[\mathrm{N}(0,\sigma^2) \geq 2] = 1-\Phi(2/\sigma)$, where $\sigma^2 = \mathrm{Var}[X(1)+X(2)] = 2\,(R_{XX}(0)+R_{XX}(1)) = 2\,(1+\mathrm{e}^{-1})$.

Task 3. $P[X(1)=1 \,|\, X(0)=0,\ X(2)=0] = P[X(0)=0,\ X(1)=1,\ X(2)=0]/P[X(0)=0,\ X(2)=0] = P[X(2)=0 \,|\, X(1)=1,\ X(0)=0]\,P[X(1)=1 \,|\, X(0)=0]/P[X(2)=0 \,|\, X(0)=0] = p_{10}p_{01}/(p_{00}p_{00}+p_{01}p_{10})$.

Task 4. By symmetry of the autocorrelationfunction $R_{XX}(\tau)$ of $X(t)$ we have $S_{XX}(\omega) = (\mathcal{F}R_{XX})(\omega) = (\mathcal{F}R_{XX}(-\cdot))(\omega) = (\mathcal{F}R_{XX})(-\omega) = S_{XX}(-\omega)$.

Task 5. As $(\mathcal{F}x')(\omega) = (\mathcal{F}(h \star x))(\omega) = H(\omega)(\mathcal{F}x)(\omega)$ and on the other hand $(\mathcal{F}x')(\omega) = \int_{-\infty}^{\infty} \mathrm{e}^{-j2\pi\omega t}x'(t)\,dt = [\mathrm{e}^{-j2\pi\omega t}x(t)]_{t=-\infty}^{t=\infty} + j2\pi\omega \int_{-\infty}^{\infty} \mathrm{e}^{-j2\pi\omega t}x(t)\,dt = j2\pi\omega \times (\mathcal{F}x)(\omega)$ we have $H(\omega) = j2\pi\omega$.

Task 6. According to theory for the Wiener filter we should use the frequency response $H(e^{j\omega}) = P_s(e^{j\omega})/(P_s(e^{j\omega})+P_w(e^{j\omega}))$, where $P_s(e^{j\omega}) = \sigma_s^2/|1+a_1\,e^{j\omega}+a_2\,e^{j2\omega}|^2$ and $P_w(e^{j\omega}) = \sigma_w^2|1+b_1\,e^{j\omega}+b_2\,e^{j2\omega}|^2$.