MVE136 Random Signals Analysis

Written exam Friday 2 November 2018 8.30-12.30

TEACHER: Patrik Albin. JOUR: Per Ljung, telephone 772 5325.

AIDS: Beta or 2 sheets (=4 pages) of hand-written notes (computer print-outs and/or xerox-copies are not allowed), but not both these aids.

GRADES: 12 (40%), 18 (60%) and 24 (80%) points for grade 3, 4 and 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Task 1. Find the autocorrelation function of the discrete time random process $\{X_n\}_{n=0}^{\infty}$ given by $X_0 = 0$ and $X_n = \sum_{k=1}^{n} \xi_k$ where $\{\xi_k\}_{k=1}^{\infty}$ are uncorrelated random variables with $E[\xi_k] = \mathrm{Var}(\xi_k) = 1$ for $k \geq 1$. **(5 points)**

Task 2. Let $\{X(t)\}_{t\geq 0}$ be a unit rate/intensity Poisson process and $\{Y(t)\}_{t\in\mathbb{R}}$ a WSS zero-mean Gaussian random process with autocorrelation function $R_{YY}(\tau) = \mathrm{e}^{-|\tau|}$ that is independent of $\{X(t)\}_{t\geq 0}$. Express the probability $Pr(X(3) \geq Y(1)+Y(2))$ in terms of (a sum of terms involving) the standard Gaussian CDF $\Phi(x) = Pr(\mathrm{N}(0,1) \leq x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \mathrm{e}^{-y^2/2}\, dy$ and the Poisson PMF $Pr(\mathrm{Po}(\lambda) = k) = \frac{\lambda^k}{k!} \mathrm{e}^{-\lambda}$. **(5 points)**

Task 3. A Markov chain $\{X(n)\}_{n=0}^{\infty}$ has two states 0 and 1 together with initial distribution $\pi(0)$ and transition matrix P given by

$$\pi(0) = [1/2 \;\; 1/2] \qquad \text{and} \qquad P = \begin{bmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{bmatrix},$$

respectively. Find the expected value $E[T]$ of the time $T = \min\{n \geq 1 : X(n) \neq X(0)\}$ the chain spends in it first state (value). [HINT: For a so called waiting time random variable ξ with probability mass function $P_\xi(k) = \Pr(\xi = k) = (1-p)^{k-1}p$ for $k = 1, 2, 3, \ldots$ it holds that $E[\xi] = 1/p$.] **(5 points)**

Task 4. Give an example of a discrete time WSS process that is not strict sense stationary. **(5 points)**

Task 5. Let $S_{XX}(f)$ be the PSD of a WSS random process $X(t)$. Prove that $S_{XX}(f)$ is real valued (has zero imaginary part). **(5 points)**

Task 6. Consider the ARMA(1,1) process

$$x[n] - 0.8\, x[n-1] = e[n] - 0.9\, e[n-1],$$

where $e[n]$ is a WSS white (noise) process with variance $\sigma_e^2 = 1$. Compute the PSD of the ARMA(1,1) process $x[n]$. **(5 points)**

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Solutions to written exam 2 November 2018

Task 1. $R_{XX}(m,n) = E[X_m X_n] = \mathrm{Cov}(X_m, X_n) + E[X_m]E[X_n] = \min(m,n) + m\,n$.

Task 2. As $Y(1)+Y(2)$ is $\mathrm{N}(0,\sigma^2)$ distributed with $\sigma^2 = 2\,(1+\mathrm{e}^{-1})$ we have $Pr(X(3) \geq Y(1)+Y(2)) = \sum_{k=0}^{\infty} Pr(\mathrm{N}(0,\sigma^2) \leq k)\, Pr(X(3)=k)$.

Task 3. $E[T] = (1/2)\cdot(1/(1/2)) + (1/2)\cdot(1/(1/3)) = 5/2$.

Task 4. Let the process $\{X(n)\}_{n=-\infty}^{\infty}$ consist of zero-mean unit variance independent (or uncorrelated) random variables that are not indentically distributed.

Task 5. By symmetry of the autocorrelationfunction $R_{XX}(\tau)$ of $X(t)$ we have $\overline{S_{XX}(f)} = \overline{(\mathcal{F}R_{XX})(f)} = (\mathcal{F}R_{XX})(-f) = (\mathcal{F}R_{XX}(-\cdot))(f) = (\mathcal{F}R_{XX})(f) = S_{XX}(f)$.

Task 6. The difference equation described in the problem formulation corresponds to a linear filter with the transfer function

$$H(z) = \frac{1-0.9\,z^{-1}}{1-0.8\,z^{-1}}.$$

The power spectral density of the output signal is therefore

$$P_{xx}(e^{j\omega}) = |H(e^{j\omega})|^2\sigma_e^2 = \frac{|1-0.9\,e^{-j\omega}|^2}{|1-0.8\,e^{-j\omega}|^2}.$$