MVE136 Random Signals Analysis

Written exam Monday 7 January 2019 2–6 pm

TEACHER: Patrik Albin. JOUR: Johannes Borgqvist, telephone 031 7725325.

AIDS: Beta or 2 sheets (=4 pages) of hand-written notes (computer print-outs and/or xerox-copies are not allowed), but not both these aids.

GRADES: 12 (40%), 18 (60%) and 24 (80%) points for grade 3, 4 and 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Task 1. Calculate $P(X(1)+X(2)=X(3))$ for a Poisson process $\{X(t)\}_{t\geq 0}$ with unit rate/intensity. **(5 points)**

Task 2. An autocorrelation function $R_{XX}(\tau)$ of a WSS process must be positive semidefinite, which is to say that $\sum_{i=1}^n \sum_{j=1}^n a_i a_j R_{XX}(t_j - t_i) \geq 0$ for all $a_1, \ldots, a_n \in \mathbb{R}$, all times $t_1, \ldots, t_n$ and all $n \in \mathbb{N}$ [as $\sum_{i=1}^n \sum_{j=1}^n a_i a_j R_{XX}(t_j - t_i) = E((\sum_{i=1}^n a_i X(t_i))^2)$]. Does this mean that $R_{XX}(\tau)$ has to be nonnegative? **(5 points)**

Task 3. Let $\{X(t)\}_{t\in\mathbb{R}}$ be a zero-mean WSS random process with autocorrelation function $R_{XX}(\tau) = \cos(\tau)$. Find the autocorrelation function $R_{YY}(s,t) = E[Y(s)Y(t)]$ of the random process $Y(t) = \int_0^t X(u)\, du$ for $t \geq 0$. **(5 points)**

Task 4. Consider a Markov chain with possible values 0 and 1 and transition matrix

$$P = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix},$$

where $\alpha, \beta \in [0,1]$ are constants. Under what additional constraints on α and β (other than that they lie in the interval $[0,1]$) does the chain possess a uniquely determined (that is, one and only one) stationary distribution π? **(5 points)**

Task 5. The input to a continuous time LTI system is a WSS continuous time random process $X(t)$ with autocorrelation function

$$R_{XX}(\tau) = \frac{A\omega_0}{\pi} \frac{\sin(\omega_0 \tau)}{\omega_0 \tau} \quad \text{for } \tau \in \mathbb{R},$$

where $A, \omega_0 > 0$ are constants. Further, the LTI system has impulse response

$$h(t) = \frac{\omega_1}{\pi} \frac{\sin(\omega_1 t)}{\omega_1 t} \quad \text{for } t \in \mathbb{R},$$

where $\omega_1 > 0$ is a constant. Find the autocorrelation function of the output $Y(t)$ from the system. **(5 points)**

Task 6. Compute the autocorrelation function $r_x[n]$ for $n = 0$ and $n = 1$ when $x[n]$ is an AR(1)-process with parameter $a_1 = 0.7$. You can assume that the input noise has variance $\sigma_e^2 = 1$. **(5 points)**

MVE136 Random Signals Analysis

Solutions to written exam 7 January 2019

Task 1. $P(X(1)+X(2)=X(3)) = P(X(1)=X(3)-X(2)) = P(\xi=\eta)$ for ξ and η independent unit mean Poisson distributed random variables so that the asked for probability is $\sum_{k=0}^{\infty} \mathrm{e}^{-2}/(k!)^2$.

Task 2. No because $\cos(\tau)$ is an autocorrelation function.

Task 3. $E[Y(s)Y(t)] = \int_{u=0}^{u=s}\int_{v=0}^{v=t} E[X(u)X(v)]\,dvdu = \int_{u=0}^{u=s}\int_{v=0}^{v=t} \cos(v-u)\,dvdu = \ldots = \cos(t-s)+1-\cos(s)-\cos(t)$.

Task 4. By basic algebraic manipulations we see that

$$\begin{cases} \pi P = \pi \\ \pi_0+\pi_1 = 1 \end{cases} \Leftrightarrow \begin{cases} \pi_0(1-\alpha)+\pi_1\beta = \pi_0 \\ \pi_0\alpha+\pi_1(1-\beta) = \pi_1 \\ \pi_0+\pi_1 = 1 \end{cases} \Leftrightarrow \begin{cases} \pi_0(1-\alpha)+\pi_1\beta = \pi_0 \\ \pi_0+\pi_1 = 1 \end{cases} \Leftrightarrow \begin{cases} \pi_0(\alpha+\beta) = \beta \\ \pi_0+\pi_1 = 1 \end{cases},$$

which in turn has a unique solution $\pi = (\frac{\beta}{\alpha+\beta} \;\; \frac{\alpha}{\alpha+\beta})$ if and only if $\alpha+\beta>0$ (whilst for $\alpha+\beta=0$ any distribution π is a solution).

Task 5. Writing $p_a(\omega)=1$ for $|\omega|<a$ and $p_a(\omega)=0$ for $|\omega|>a$ we have $S_{XX}(\omega) = A\,p_{\omega_0}(\omega)$ and $H(\omega)=p_{\omega_1}(\omega)$ so that $S_{YY}(f) = |H(f)|^2 S_{XX}(f) = A\,p_{\omega_0}(\omega)\,p_{\omega_1}(\omega) = A\,p_{\min\{\omega_0,\omega_1\}}(\omega)$ giving

$$R_{YY}(\tau) = \frac{A\min\{\omega_0,\omega_1\}}{\pi}\,\frac{\sin(\min\{\omega_0,\omega_1\}\tau)}{\min\{\omega_0,\omega_1\}\tau} \quad \text{for } \tau\in\mathbb{R}.$$

Task 6. We can use the Yule-Walker equations to find $r_x[0]$ and $r_x[1]$. Since it is simple, this solution will start with a derivation of Yule-Walker: If we multiply both sides of the equation

$$x[n]+0.7x[n-1]=e[n]$$

with $x[n-k]$ and take expectations we get

$$\underbrace{E\{x[n-k](x[n]+0.7x[n-1])\}}_{r_x[k]+0.7r_x[k-1]} = \underbrace{E\{x[n-k]e[n]\}}_{\delta[k]} \quad \text{for } k\geq 0.$$

Using this equation for $k=0$ and $k=1$ we get the matrix equation

$$\begin{bmatrix} 1 & 0.7 \\ 0.7 & 1 \end{bmatrix} \begin{bmatrix} r_x[0] \\ r_x[1] \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Solving this matrix equation we get $r_x[0] = 1/0.51 \approx 1.96$ and $r_x[1] = -0.7r_x[0] \approx 1.37$.