Lecture notes MVE136 Fall 2019. Part 3: LTI Systems

10. Power Spectral Densities

Definition 10.1. Two WSS processes $X(t)$ and $Y(t)$ are called *jointly WSS* if $R_{XY}(t, t+\tau) = R_{XY}(\tau)$ does not depend on t.

Example 10.1. If $X(t)$ is WSS and $Y(t) = X(-t)$ then $R_{XY}(t, t+\tau) = E(X(t)X(-t-\tau)) = R_{XX}(2t+\tau)$ which depends on t so that $X(t)$ and $Y(t)$ are not jointly WSS unless $R_{XX}(\tau) = R_{XX}(0)$ is constant.

Definition 10.2. The *power spectral density (PSD)* S_{XX} of a WSS process $X(t)$ is the Fourier transform of the ACF

$$S_{XX}(f) = (\mathcal{F}R_{XX})(f) = \begin{cases} \int\limits_{-\infty}^{\infty} \mathrm{e}^{-j2\pi f\tau} R_{XX}(\tau)\, d\tau & \text{for } f \in \mathbb{R} \text{ in continuous time,} \\ \sum\limits_{k=-\infty}^{\infty} \mathrm{e}^{-j2\pi fk} R_{XX}(k) & \text{for } f \in [-\frac{1}{2}, \frac{1}{2}] \text{ in discrete time.} \end{cases}$$

The *crosspectraldensity* S_{XY} between two jointly WSS processes $X(t)$ and $Y(t)$ is defined $S_{XY}(f) = (\mathcal{F}R_{XY})(f)$.

Corollary 10.3.

$$R_{XX}(\tau) = (\mathcal{F}^{-1}S_{XX})(\tau) = \begin{cases} \int_{-\infty}^{\infty} \mathrm{e}^{j2\pi f\tau} S_{XX}(f)\, df & \text{in continuous time,} \\ \int_{-1/2}^{1/2} \mathrm{e}^{j2\pi f\tau} S_{XX}(f)\, df & \text{in discrete time.} \end{cases}$$

$$R_{XY}(\tau) = (\mathcal{F}^{-1}S_{XY})(\tau) = \begin{cases} \int_{-\infty}^{\infty} \mathrm{e}^{j2\pi f\tau} S_{XY}(f)\, df & \text{in continuous time,} \\ \int_{-1/2}^{1/2} \mathrm{e}^{j2\pi f\tau} S_{XY}(f)\, df & \text{in discrete time.} \end{cases}$$

Proof. Fourier inversion formula and Fourier series expansion, respectively. □

The *bandwidth* is a certain measure of the width of the graph of the PSD. There are several different ways to define bandwidth and a common one is the 3dB-bandwidth which is the width of the zone were the PSD is at least half as big as its maximal value.

We use the notation $\overline{z} = \Re(z) - j\,\Im(z)$ for the complex conjugate of $z \in \mathbb{C}$.

In the following theorem Property 3 is very deep and very difficult to prove:

Theorem 10.4. (PROPERTIES OF PSD) For $X(t)$ WSS we have

1. $S_{XX}(f)$ is real,
2. $S_{XX}(f) = S_{XX}(-f)$,
3. $S_{XX}(f) \geq 0$,
4. $$E(X(t)^2) = R_{XX}(0) = \begin{cases} \int_{-\infty}^{\infty} S_{XX}(f)\, df & \text{in continuous time,} \\ \int_{-1/2}^{1/2} S_{XX}(f)\, df & \text{in discrete time.} \end{cases}$$
5. For $X(t)$ and $Y(t)$ jointly WSS we have $$E(X(t)Y(t)) = R_{XY}(0) = \begin{cases} \int_{-\infty}^{\infty} S_{XY}(f)\, df & \text{in continuous time,} \\ \int_{-1/2}^{1/2} S_{XY}(f)\, df & \text{in discrete time.} \end{cases}$$

Proof. (1)-(2) By symmetry of R_{XX} we have $\overline{S_{XX}(f)} = S_{XX}(-f) = (\mathcal{F}(R_{XX}(-\cdot)))(f) = (\mathcal{F}R_{XX})(f)$ where the middle equality is by change of variable in the Fourier transform.

(4)-(5) Take $\tau = 0$ in the inversion formulas of the previous corollary. □

Definition 10.5. *White noise* is a WSS zero-mean process $N(t)$ with constant PSD $S_{NN}(f) = N_0/2$ for some $N_0 > 0$ so that $R_{NN}(\tau) = (N_0/2)\,\delta(\tau)$.

Above δ is Kronecker's δ-function given by $\delta(0) = 1$ and $\delta(k) = 0$ for $k \in \mathbb{Z} \setminus \{0\}$ in discrete time and Dirac's δ-function given by $\int_{-\infty}^{\infty} \delta(t)g(t)\, dt = g(0)$ in continuous time.

Sometimes white noise is also required to be Gaussian. White noise in continuous time does not exist but is widely used in engineering math anyway. In discrete time white noise is a sequence of zero-mean uncorrelated r.v.'s with common variance $N_0/2$.

Example 10.2. The ACF $R_{XX}(\tau) = \mathrm{e}^{-\alpha|\tau|}$ for $\tau \in \mathbb{R}$ with $\alpha > 0$ has PSD

$$S_{XX}(f) = \int_0^{\infty} \mathrm{e}^{-j2\pi f\tau}\mathrm{e}^{-\alpha\tau}\, d\tau + \int_{-\infty}^{0} \mathrm{e}^{-j2\pi f\tau}\mathrm{e}^{\alpha\tau}\, d\tau = \frac{1}{\alpha+j2\pi f} + \frac{1}{\alpha-j2\pi f} = \frac{2\,\alpha}{\alpha^2+4\pi^2 f^2}.$$

From this and the fact that Fourier transformation and Fourier inversion for symmetric functions are the same mathematical operations in continuous time we conclude that the ACF $R_{XX}(\tau) = 2\,\alpha/(\alpha^2 + 4\pi^2\tau^2)$ has PSD $S_{XX}(f) = \mathrm{e}^{-\alpha|f|}$.

The previous example illustrate the important fact that when calculating the PSD of an ACF one do always earn another Fourier transform calculation for free (unless the

ACF and PSD agree which happens only for the Gaussian bell function).

Example 8.3. (CONTINUED) For $X(t) = a\sin(\omega t + \Theta)$ for $t \in \mathbb{R}$ where Θ is uniformly distributed over $[0, 2\pi]$ and $\omega, a \in \mathbb{R}$ we have $R_{XX}(\tau) = \frac{1}{2}\, a^2 \cos(\omega\tau)$ which gives $S_{XX}(f) = \frac{1}{4}\, a^2[\delta(f - \omega/(2\pi)) + \delta(f + \omega/(2\pi))]$ by the "backway method" to figure what $S_{XX}(f)$ must be to produce the given ACF $R_{XX}(\tau) = (\mathcal{F}^{-1} S_{XX})(\tau)$ by Fourier inversion rather than by Fourier transforming R_{XX}.

Example 10.3. (AR(1)-PROCESS) Consider a discrete time zero-mean WSS process $Y[k]$ that satisfies $Y[k] = a\,Y[k{-}1] + b\,e[k]$ where $a \in (-1, 1)$, $b \in \mathbb{R}$ and $e[k]$ is unit variance white noise with $\mathrm{Cov}(e[k], Y[k{-}\ell]) = 0$ for $\ell \geq 1$. We have

$$\begin{aligned} R_{YY}(0) &= E(Y[k]^2) = E((a\,Y[k{-}1] + b\,e[k])^2) \\ &= a^2\, E(Y[k{-}1]^2) + 2\,ab\, E(Y[k{-}1]e[k]) + b^2\, E(e[k]^2) = a^2 R_{YY}(0) + b^2 \end{aligned}$$

so that $R_{YY}(0) = b^2/(1 - a^2)$. For $k \geq 1$ we further have

$$R_{YY}(k) = E(Y[\ell]Y[\ell{+}k]) = E\big(Y[\ell]\,(a\,Y[k{+}\ell{-}1] + b\,e[k{+}\ell])\big) = a\,R_{YY}(k{-}1)$$

which is a difference equation with solution $R_{YY}(k) = a^{|k|} b^2/(1 - a^2)$. And so

$$S_{YY}(f) = \sum_{k=0}^{\infty} \frac{a^k b^2}{1 - a^2}\, \mathrm{e}^{-j2\pi k f} + \sum_{k=0}^{\infty} \frac{a^k b^2}{1 - a^2}\, \mathrm{e}^{j2\pi k f} - \frac{b^2}{1 - a^2} = \ldots = \frac{b^2}{|1 - a\,\mathrm{e}^{-j2\pi f}|^2}.$$

Non-parametric spectral estimation

Given an observation $\{X(t)\}_{t \in [-t_0, t_0]}$ of a continous time WSS process $X(t)$ the natural estimate (based on ergodicity in the autocorrelation) of its ACF R_{XX} if unknown is

$$\hat{R}_{XX}(\tau) = \begin{cases} \dfrac{1}{2t_0 - \tau} \displaystyle\int_{-t_0+|\tau|/2}^{t_0-|\tau|/2} X(t - |\tau|/2)X(t + |\tau|/2)\, dt & \text{for } |\tau| \leq 2t_0, \\ 0 & \text{for } |\tau| > 2t_0. \end{cases}$$

It is suitable to damp $R_{XX}(\tau)$-estimates for big $|\tau|$ close to $2t_0$ with a *windowing function* $|w(\tau)| \leq 1$ since such estimates are uncertain because based on few data. The resulting windowed estimate is $\hat{R}^{(w)}_{XX}(\tau) = w(\tau)\hat{R}_{XX}(\tau)$. A common window choice is

$$w(\tau) = \mathrm{tri}(\tau/(2t_0)) = \begin{cases} 1 - |\tau|/(2t_0) & \text{for } |\tau| \leq 2t_0, \\ 0 & \text{for } |\tau| > 2t_0. \end{cases}$$

This yields the most commonly used ACF estimate

$$\hat{R}^{(\mathrm{tri})}_{XX}(\tau) = \mathrm{tri}(\tau/(2t_0))\hat{R}_{XX}(\tau) = \begin{cases} \dfrac{1}{2t_0} \displaystyle\int_{-t_0+|\tau|/2}^{t_0-|\tau|/2} X(t-|\tau|/2)X(t+|\tau|/2)\,dt & \text{for } |\tau|\leq 2t_0, \\ 0 & \text{for } |\tau|>2t_0. \end{cases}$$

To estimate $S_{XX}(f)$ if unknown one uses $\hat{S}_{XX}(f) = (\mathcal{F}\hat{R}_{XX})(f)$. The most commonly used PSD estimate thus becomes $\hat{S}^{(\mathrm{tri})}_{XX}(f) = (\mathcal{F}\hat{R}^{(\mathrm{tri})}_{XX})(f)$.

We write $X_{t_0}(t)$ for the process that is $X(t)$ for $t \in [-t_0, t_0]$ and 0 otherwise.

Theorem 10.6. (The Periodogram) $\hat{S}^{(\mathrm{tri})}_{XX}(f) = |(\mathcal{F}X_{t_0})(f)|^2/(2t_0)$.

Proof.

$$\hat{S}^{(\mathrm{tri})}_{XX}(f) = \frac{1}{2t_0}\int_{-\infty}^{\infty} \mathrm{e}^{-j2\pi\tau f}\Big(\int_{-t_0+|\tau|/2}^{t_0-|\tau|/2} X(t-|\tau|/2)X(t+|\tau|/2)\,dt\Big)d\tau$$
$$= \frac{1}{2t_0}\int_{-\infty}^{\infty} \mathrm{e}^{-j2\pi\tau f}\Big(\int_{-t_0}^{t_0} X_{t_0}(u)X_{t_0}(u-\tau)\,du\Big)d\tau = \frac{1}{2t_0}(\mathcal{F}(X_{t_0}\star(X_{t_0}(-\cdot)))(f). \quad \square$$

Parametric spectral estimation

As opposed to non-parametric estimation where no detailed theoretical information about the process is available there is parametric estimation where data is assumed to come from a fully theoretically specified process except that one or a few parameters have unkown value(s). We illustrate parametric spectral estimation by an example:

Example 10.3. (Continued) For observed data $\{Y[k]\}_{k=0}^{n}$ of the AR(1)-process $Y[k] = a\,Y[k-1] + b\,e[k]$ with the parameters $a \in (-1,1)$ and $b^2 > 0$ unknown we may use the ACF $R_{YY}(k) = a^{|k|}b^2/(1-a^2)$ to estimate a and b^2 by solving

$$\frac{\hat{b}^2}{1-\hat{a}^2} = \hat{R}_{YY}(0) = \frac{1}{n+1}\sum_{k=0}^{n} Y[k]^2 \quad \text{and} \quad \frac{\hat{a}\,\hat{b}^2}{1-\hat{a}^2} = \hat{R}_{YY}(1) = \frac{1}{n}\sum_{k=1}^{n} Y[k]Y[k-1]$$

for $\hat{a}$ and $\hat{b}^2$ and then estimate $\hat{S}_{YY}(f) = \hat{b}^2/|1-\hat{a}\,\mathrm{e}^{-j2\pi f}|^2$.

11. Linear Timeinvariant Systems

Definition 11.1. A *linear time invariant (LTI) system* with insignal $x(t)$ and outsignal $y(t) = (Tx)(t)$ satisfies

1. $(T(\alpha\,x_1 + \beta\,x_2))(t) = \alpha\,(Tx_1)(t) + \beta\,(Tx_2)(t)$ for $\alpha, \beta \in \mathbb{R}$,
2. $(T(x(\cdot - t_0)))(t) = (Tx)(t-t_0)$ for $t_0 \in \mathbb{R}$.

An LTI system either has discrete time $t \in \mathbb{Z}$ or continuous time $t \in \mathbb{R}$.

Definition 11.2. The *impulse response* h of an LTI system is $h(t) = (T\delta)(t)$.

Theorem 11.3. For an LTI system we have

$$(Tx)(t) = (h\star x)(t) = \begin{cases} \int_{-\infty}^{\infty} h(t-u)\,x(u)\,du = \int_{-\infty}^{\infty} h(u)\,x(t-u)\,du & \text{in continuous time,} \\ \sum_{k=-\infty}^{\infty} h(t-k)\,x(k) = \sum_{k=-\infty}^{\infty} h(k)\,x(t-k) & \text{in discrete time.} \end{cases}$$

Proof. (Discrete time.) As $x(t) = \sum_{k=-\infty}^{\infty} x(k)\,\delta(t-k)$ we have

$$(Tx)(t) = \big(T\big(\sum_{k=-\infty}^{\infty} x(k)\,\delta(\cdot - k)\big)\big)(t) = \sum_{k=-\infty}^{\infty} x(k)\,(T\delta(\cdot - k))(t) = \sum_{k=-\infty}^{\infty} x(k)\,(T\delta)(t-k). \qquad \square$$

Theorem 11.4. An WSS insignal $X(t)$ to an LTI system is jointly WSS with the outsignal $Y(t)$ and

$$\mu_Y = h\star\mu_X, \quad R_{XY}(\tau) = (h\star R_{XX})(\tau) \quad \text{and} \quad R_{YY}(\tau) = ((h(-\cdot))\star h\star R_{XX})(\tau).$$

Proof. (Continuous time.)

$$\mu_Y(t) = E(h\star X(t)) = E\big(\int_{-\infty}^{\infty} h(u)X(t-u)\,du\big) = \int_{-\infty}^{\infty} h(u)\,E(X(t-u))\,du = \int_{-\infty}^{\infty} h(u)\,\mu_X\,du,$$

$$\begin{aligned} R_{XY}(t,t+\tau) &= E\big(X(t)\int_{-\infty}^{\infty} h(u)X(t+\tau-u)\,du\big) \\ &= \int_{-\infty}^{\infty} h(u)\,E(X(t)X(t+\tau-u))\,du = (h\star R_{XX})(\tau), \end{aligned}$$

$$\begin{aligned} R_{YY}(t,t+\tau) &= E\big[\big(\int_{-\infty}^{\infty} h(u)X(t-u)\,du\big)\big(\int_{-\infty}^{\infty} h(v)X(t+\tau-v)\,dv\big)\big] \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(u)h(v)\,R_{XX}(\tau+u-v)\,dudv = ((h(-\cdot))\star h\star R_{XX})(\tau). \qquad \square \end{aligned}$$

Definition 11.5. The *transfer function* H of an LTI system is $H(f) = (\mathcal{F}h)(f)$.

Corollary 11.6. $S_{XY}(f) = H(f)\,S_{XX}(f)$, $S_{YX}(f) = \overline{H(f)}\,S_{XX}(f)$ and $S_{YY}(f)$ $= |H(f)|^2 S_{XX}(f)$.

Proof. The first identity follows using that $(\mathcal{F}(g\star h))(f) = (\mathcal{F}g)(f)\,(\mathcal{F}h)(f)$. As $(\mathcal{F}(g(-\cdot)))(f) = \overline{(\mathcal{F}g)(f)}$ for realvalued functions g the second identity follows using that $R_{YX}(\tau) = R_{XY}(-\tau)$ so $S_{YX}(f) = \overline{(\mathcal{F}R_{XY})(f)}$ and the third identity similary. $\square$

Example 10.3. (Continued) Viewing the AR(1)-process $Y[k] = a\,Y[k-1] + b\,e[k]$ as an LTI system with input e and output Y we have

$$H(f)\,(\mathcal{F}e)(f) = (\mathcal{F}Y)(f) = \sum_{k=-\infty}^{\infty} \mathrm{e}^{-j2\pi kf} Y[k] = a \sum_{k=-\infty}^{\infty} \mathrm{e}^{-j2\pi kf} Y[k-1] + b\,(\mathcal{F}e)(f)$$

$$= a\,\mathrm{e}^{-j2\pi kf}(\mathcal{F}Y)(f) + b\,(\mathcal{F}e)(f) = a\,\mathrm{e}^{-j2\pi kf} H(f)\,(\mathcal{F}e)(f) + b\,(\mathcal{F}e)(f)$$

so that $H(f) = b/(1 - a\,\mathrm{e}^{-j2\pi kf})$. From this we recover our earlier finding

$$S_{YY}(f) = |H(f)|^2 S_{ee}(f) = |H(f)|^2 = b^2/|1 - a\,\mathrm{e}^{-j2\pi f}|^2.$$

The matched filter

In a digital communication system 1 is represented by sending a deterministic signal $s(t)$ while 0 i represented by sending 0. The signal is sent on a noisy channel with additive white noise $N(t)$ so that the received signal is $X(t) = s(t) + N(t)$ if 1 is sent and $X(t) = N(t)$ if 0 is sent.

To obtain best signal detection (of whether 1 or 0 is sent) the received signal is processed through an LTI system with impulse response h designed to maximize the signal to noise ratio (SNR) $E((h \star s)(t)^2)/E((h \star N)(t)^2)$ at a certain detection time t.

Using Property 4 of PSD together with the Planecherel identity $\int |(\mathcal{F}g)(f)|^2\,df = \int |g(x)|^2\,dx$ and Cauchy-Schwarz inequality for integrals $(\int g(x)h(x)\,dx)^2 \leq (\int g(x)^2\,dx)$ $(\int h(x)^2\,dx)$ we see that in continuous time

$$\mathrm{SNR} = \frac{(h \star s)(t)^2}{\int_{-\infty}^{\infty} |H(f)|^2 (N_0/2)\,df} = \frac{2}{N_0}\,\frac{[\int_{-\infty}^{\infty} h(u)s(t-u)\,du]^2}{\int_{-\infty}^{\infty} h(u)^2\,du} \leq \frac{2}{N_0}\int_{-\infty}^{\infty} s(t-u)^2\,du$$

where the right-hand side is independent of h. As we have equality above for $h(u) = s(t-u)$ this is the h we seek. We have proved the following theorem in continuous time:

Theorem 11.7. (Matched Filter) $h(u) = s(t-u)$.

The Wiener filter

A WSS process $X(t)$ with PSD $S_{XX}(f)$ is sent on a noisy channel where an independent zero-mean WSS noise $N(t)$ with PSD $S_{NN}(f)$ is added.

The recived signal $Y(t) = X(t) + N(t)$ is processed through an LTI system with impulse response h and transfer function H designed to minimize the mean-square distance $E((X(t) - (h \star Y)(t))^2)$ between the sent signal and the processed recived signal.

Using Properties 4 and 5 of spectraldensities we see that in continuous time

$$
\begin{aligned}
& E(((h\star Y)(t) - X(t))^2) \\
&= E\big(((h\star X)(t) + (h\star N)(t) - X(t))^2\big) \\
&= E\big((h\star X)(t)^2 + (h\star N)(t) + X(t)^2 + 2\,(h\star X)(t)\,(h\star N)(t) - 2\,(h\star X)(t)\,X(t) - 2\,(h\star N)(t)\,X(t)\big) \\
&= \int_{-\infty}^{\infty} \big(|H(f)|^2 S_{XX}(f) + |H(f)|^2 S_{NN}(f) + S_{XX}(f) - 2\,H(f)\,S_{XX}(f)\big)\,df.
\end{aligned}
$$

Minima is for the derivative of the integrand $2\,H(f)\,(S_{XX}(f) + S_{NN}(f)) - 2\,S_{XX}(f)$ wrt. $H(f)$ equals zero. We have proved the following theorem in continuous time:

Theorem 11.8. (Wiener Filter) $H(f) = S_{XX}(f)/(S_{XX}(f) + S_{NN}(f))$.

Exercise 1. In the derivation we replaced $|H(f)|^2$ with $H(f)^2$ - justify this step.