MVE136 Random Signals Analysis

Written exam Monday 19 August 2019 2-6 pm

TEACHER: Patrik Albin. JOUR: Juan Inda, telephone 772 5325.

AIDS: Beta or 2 sheets (=4 pages) of hand-written notes (computer print-outs and/or xerox-copies are not allowed), but not both these aids.

GRADES: 12 (40%), 18 (60%) and 24 (80%) points for grade 3, 4 and 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Task 1. Find the probability $Pr(X(1)+X(2)+X(3) > 6)$ for a continuous time WSS Gaussian process $X(t)$ with mean $\mu_X = 1$ and autocorrelation function $R_{XX}(\tau) = \mathrm{e}^{-|\tau|} + 1$ for $\tau \in \mathbb{R}$. **(5 poäng)**

Task 2. Find the probability $Pr(X(1) + X(2) > 3)$ for a Poisson process with rate 1. **(5 poäng)**

Task 3. Give an example of a WSS random process that is not strict sense stationary. **(5 poäng)**

Task 4. Calculate $E(X[n]X[n+1])$ for non-negative integers n when $\{X[n], n \geq 0\}$ is a Markov chain with state space (/possible values) S, initial distribution $\pi(0)$ and transition probability matrix P, respectively, given by

$$S = \{0, 1\}, \qquad \pi(0) = [1/2 \;\; 1/2] \qquad \text{and} \qquad P = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}. \qquad \textbf{(5 points)}$$

Task 5. The sum $X(t) = S(t)+N(t)$ of a continuous time signal process $S(t)$ with PSD $S_{SS}(f) = 2/(2+(2\pi f)^2)$ and an independent (of $S(t)$) continuous time noise process $N(t)$ with PSD $S_{NN}(f) = 1$ is input to an LTI filter. Determine the impulse response of the filter that produces an output $Y(t)$ from the filter that minimizes the mean-square signal-recovering error $E[(S(t) - Y(t))^2]$. Also, find the PSD of the output $Y(t)$ when the filter has this impulse response. **(5 points)**

Task 6. Prediction is an important tool in many contexts, including decision theory, planning and control. Suppose that we wish to predict $x[n+1]$ given $x[n]$ and $x[n-1]$, where $x[n]$ is an MA(2)-process

$$x[n] = e[n] + 0.5\, e[n-1] + 0.2\, e[n-2] \quad \text{for } n \in \mathbb{N}.$$

The input noise $e[n]$ is assumed to be a wide sense stationary zero mean white noise process with variance $E[e[n]^2] = 1$ and autocorrelation function $r_e[k] = E[e[n]e[n+k]] = 0$ for $k \neq 0$. We will further assume that we seek a linear estimator

$$\hat{x}[n+1] = h_0\, x[n] + h_1\, x[n-1].$$

Find the coefficients h_0 and h_1 that minimize $E[(x[n+1] - \hat{x}[n+1])^2]$. **(5 poäng)**

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Solutions to written exam 19 August 2019

Task 1. As $X(1)+X(2)+X(3)$ is $N(m,\sigma^2)$-distributed we have $Pr(X(1)+X(2)+X(3) > 6) = Pr(N(m,\sigma^2) > 6) = 1-\Phi((6-m)/\sigma)$, where $m = E[X(1)+X(2)+X(3)] = 3$ and $\sigma^2 = \mathrm{Var}(X(1)+X(2)+X(3)) = 3\,C_{XX}(0)+4\,C_{XX}(1)+2\,C_{XX}(2) = 3+4\,\mathrm{e}^{-1}+2\,\mathrm{e}^{-2}$ [using that $C_{XX}(\tau) = R_{XX}(\tau)-\mu_X^2 = \mathrm{e}^{-|\tau|}$].

Task 2. As $X(1)+X(2) = (X(2)-X(1))+2\,X(1)$ where $X(2)-X(1)$ and $X(1)$ are independent Po(1)-distributed we have $Pr(X(1)+X(2) > 3) = Pr((X(2)-X(1)) + 2\,X(1) > 3) = Pr(X(1) \geq 2) + Pr(X(1)=1,\, X(2)-X(1) > 1) + Pr(X(1)=0,\, X(2)-X(1) > 3) = (1-\mathrm{e}^{-1}-\mathrm{e}^{-1}) + \mathrm{e}^{-1}(1-\mathrm{e}^{-1}-\mathrm{e}^{-1}) + \mathrm{e}^{-1}(1-\mathrm{e}^{-1}-\mathrm{e}^{-1}-\frac{1}{2}\mathrm{e}^{-1}-\frac{1}{6}\mathrm{e}^{-1})$.

Task 3. For example, the process $\{X(t)\}_{t\in\mathbb{Z}}$ made up of independent random variables that are $\mathrm{N}(0,1)$-distributed for t even and that have a discrete Rademacher distribution with PMF $P_{X(t)}(-1) = P_{X(t)}(1) = 1/2$ for t odd, as this process is zero-mean with autocorrelation function $R_{XX}(\tau) = \delta(\tau)$, but clearly is not strict sense stationary.

Task 4. As the only possible values of the random variable $X[n]X[n+1]$ are 0 and 1, we have $E(X[n]X[n+1]) = 0\cdot P(X[n]X[n+1]=0) + 1\cdot P(X[n]X[n+1]=1) = P(X[n]X[n+1]=1) = P(X[n]=1,\, X[n+1]=1) = \pi(n)_1 P_{11} = \pi(n)_1(1/2)$. Noting that $\pi(0) = \pi$ is in fact a stationary distribution for the chain we have $\pi(n) = \pi(0) = \pi$, so that $\pi(n)_1 = \pi(0)_1 = 1/2$ and $E(X[n]X[n+1]) = 1/4$. (The latter result is in fact also more or less obviously true already from the beginning without calculations from symmetry considerations ... .)

Task 5. We are looking for the impulse response of the Wiener filter with transfer function $H(f) = S_{SS}(f)/(S_{SS}(f)+S_{NN}(f)) = 2/(4+(2\pi f)^2)$ so that the impulse response (see Table E.1 in the book of Miller and Childers) is $h(t) = 2\,\mathrm{e}^{-2|t|}$. Further, we have

$$S_{YY}(f) = |H(f)|^2 S_{XX}(f) = \frac{S_{SS}(f)^2}{S_{SS}(f)+S_{NN}(f)} = \frac{4}{(4+(2\pi f)^2)\,(2+(2\pi f)^2)}.$$

Task 6. Let us first formulate this problem in terms of our standard notation for Wiener filtering: Our quantity of interest is usually denoted $d[n]$ whereas our estimator is denoted $\hat{d}[n]$. In this problem, we have $d[n] = x[n+1]$ and $\hat{d}[n] = h_0\,x[n] + h_1\,x[n-1]$. In terms of this notation, the Wiener-Hopf equations can be expressed as

$$\begin{cases} h_0\, r_x[0] + h_1\, r_x[1] = r_{dx}[0] \\ h_0\, r_x[1] + h_1\, r_x[0] = r_{dx}[1] \end{cases},$$

which can also be written on matrix form

$$\begin{bmatrix} r_x[0] & r_x[1] \\ r_x[1] & r_x[0] \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} = \begin{bmatrix} r_{dx}[0] \\ r_{dx}[1] \end{bmatrix}. \tag{1}$$

For this problem, we note that $r_{dx}[0] = E[d[n]x[n]] = E[x[n+1]x[n]] = r_x[1]$ and $r_{dx}[1] = E[d[n]x[n-1]] = E[x[n+1]x[n-1]] = r_x[2]$ so that it is therefore sufficient to compute $r_x[k]$ for $k = 0$, 1 and 2 before we can solve for h_0 and h_1.

Let us try to derive a general expression for $r_x[k]$: Given our expression for $x[n]$ it holds that

$$\begin{aligned} r_x[k] &= E[x[n]x[n-k]] \\ &= E\big[(e[n] + 0.5\, e[n-1] + 0.2\, e[n-2])\,(e[n-k] + 0.5\, e[n-1-k] + 0.2\, e[n-2-k])\big] \\ &= \begin{cases} 0 & \text{if } |k| > 2 \\ 0.2 & \text{if } |k| = 2 \\ 0.5 + 0.2 \cdot 0.5 = 0.51 & \text{if } |k| = 1 \\ 1 + 0.5 + 0.2 = 1.7 & \text{if } k = 0 \end{cases}. \end{aligned} \tag{2}$$

By combining (1) with (2), we obtain the matrix equation

$$\begin{bmatrix} 1.7 & 0.51 \\ 0.51 & 1.7 \end{bmatrix} \begin{bmatrix} h_0 \\ h_1 \end{bmatrix} = \begin{bmatrix} 0.51 \\ 0.2 \end{bmatrix},$$

from which we can find the solutions $h_0 \approx 0.29$ and $h_1 \approx 0.03$.