MVE136 Random Signals Analysis

Written exam Friday 1 November 8.30–12.30

TEACHER: Patrik Albin 031 7723512. JOUR: Johan Tykesson, telephone 070 3182096.

AIDS: Beta or 2 sheets (=4 pages) of hand-written notes (computer print-outs and/or xerox-copies are not allowed), but not both these aids.

GRADES: 12 (40%), 18 (60%) and 24 (80%) points for grade 3, 4 and 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Task 1. Two jointly WSS jointly Gaussian continuous time random processes $X(t)$ and $Y(t)$, $t \in \mathbb{R}$, have common mean $\mu_X = \mu_Y = 0$, common autocorrelation function $R_{XX}(\tau) = R_{YY}(\tau) = \mathrm{e}^{-|\tau|}$ and crosscorrelation function $R_{XY}(\tau) = \frac{1}{2}\,\mathrm{e}^{-|\tau|}$ for $\tau \in \mathbb{R}$. Calculate $P(X(1)+Y(2) > 3)$. [HINT: For $X(t)$ and $Y(t)$ jointly Gaussian each linear combination $\sum_{i=1}^{m} a_i X(s_i) + \sum_{j=1}^{n} b_j Y(t_j)$ is normal distributed.] **(5 points)**

Task 2. Let $X(t)$, $t \geq 0$, be a Poisson process with arrival rate (intensity) 1. Calculate $P(X(1) \geq 1,\ X(2) \leq 2,\ X(3) \geq 1)$. **(5 points)**

Task 3. A Markov chain $X(n)$, $n = 0, 1, 2, \ldots$, has states (possible values) 0 and 1 with starting distribution $\pi(0) = (1\ \ 0)$ and transition matrix $P = \begin{pmatrix} 1/3 & 2/3 \\ 2/3 & 1/3 \end{pmatrix}$. Calculate $R_{XX}(2,4) = E(X(2)X(4))$. **(5 points)**

Task 4. Find a WSS continuous time random process $X(t)$, $t \in \mathbb{R}$, that has autocorrelation function $R_{XX}(\tau) = 1$ for $\tau \in \mathbb{R}$. Also calculate the PSD $S_{XX}(f)$. **(5 points)**

Task 5. In a digital communication system 1 is represented by sending an analogue continuous time deterministic signal $s(t)$, $t \in \mathbb{R}$, while 0 i represented by sending 0. The signal is sent on a noisy channel with a zero-mean Gaussian WSS continuous time white noise process $N(t)$, $t \in \mathbb{R}$, having $R_{NN}(\tau) = (N_0/2)\,\delta(\tau)$ and $S_{NN}(f) = N_0/2$. Hence the recived signal is $X(t) = s(t) + N(t)$ if 1 is sent and $X(t) = N(t)$ if 0 is sent.

To obtain best signal detection at detection time t_0 the recived signal $X(t)$ is processed through a matched filter LTI system with impulse response $h(u) = s(t_0 - u)$ for $u \in \mathbb{R}$ designed to maximize the signal to noise ratio

$$\begin{aligned}\mathrm{SNR} &= \frac{(h \star s)(t_0)^2}{\mathrm{Var}((h \star N)(t_0))} = \frac{(h \star s)(t_0)^2}{E((h \star N)(t_0)^2)} = \frac{(\int_{-\infty}^{\infty} s(t_0 - u)^2\,du)^2}{\int_{-\infty}^{\infty} |H(f)|^2 S_{NN}(f)\,df} \\ &= \frac{2}{N_0}\,\frac{(\int_{-\infty}^{\infty} s(u)^2\,du)^2}{\int_{-\infty}^{\infty} |H(f)|^2\,df} = \frac{2}{N_0}\,\frac{(\int_{-\infty}^{\infty} s(u)^2\,du)^2}{\int_{-\infty}^{\infty} h(u)^2\,du} = \frac{2}{N_0}\int_{-\infty}^{\infty} s(u)^2\,du\end{aligned}$$

At detection it is decided that 1 was sent if $(h \star X)(t_0) > \frac{1}{2}(h \star s)(t_0)$ while it is decided that 0 was sent if $(h \star X)(t_0) \leq \frac{1}{2}(h \star s)(t_0)$. Find the probabilities for erroneous detection P(decision 1 when 0 was sent) and P(decision 0 when 1 was sent). [HINT: As $(h \star N)(t_0)$ is normal distributed so is $(h \star X)(t_0)$.] **(5 points)**

Task 6. Shortly describe Blackman-Tukey's method to improve on PSD estimation. What improvment does it bring and what is a possible drawback? **(5 points)**

MVE136 Random Signals Analysis

Solutions to written exam 1 November 2019

Task 1. As $X(1)+Y(2)$ is $\mathrm{N}(\mu,\sigma^2)$ with $\mu=\mu_X+\mu_Y=0$ and $\sigma^2=\mathrm{Var}\{X(1)+Y(2)\}$ $=E((X(1)+Y(2))^2)=R_{XX}(0)+2R_{XY}(1)+R_{YY}(0)=2+\mathrm{e}^{-1}$ we have $P(X(1)+Y(2)$ $>3)=1-\Phi\big(\frac{3-\mu}{\sigma}\big)=1-\Phi\big(\frac{3}{\sqrt{2+\mathrm{e}^{-1}}}\big)$.

Task 2. The possible values of $(X(1),X(2)-X(1))$ are $(1,0)$, $(1,1)$ and $(2,0)$ with probabilities $P(\mathrm{Po}(1)=1)\cdot P(\mathrm{Po}(1)=0)=\mathrm{e}^{-2}$, $P(\mathrm{Po}(1)=1)\cdot P(\mathrm{Po}(1)=1)=\mathrm{e}^{-2}$ and $P(\mathrm{Po}(1)=2)\cdot P(\mathrm{Po}(1)=0)=\frac{1}{2}\,\mathrm{e}^{-2}$, respectively, so that $P(X(1)\geq 1,\ X(2)\leq 2,\ X(3)\geq 1)=P(X(1)\geq 1,\ X(2)\leq 2)=\frac{5}{2}\,\mathrm{e}^{-2}$.

Task 3. As $P^{(2)}=P^2=\begin{pmatrix}5/9 & 4/9\\ 4/9 & 5/9\end{pmatrix}$ we have $E(X(2)X(4))=1\cdot 1\cdot P(X(2)=1,\ X(4)=1)$ $=(\pi(2))_1\,(P^{(2)})_{11}=(\pi(0)\,P^2)_1\,(P^2)_{11}=\frac{4}{9}\cdot\frac{5}{9}=\frac{20}{81}\approx\frac{1}{4}$.

Task 4. Let Y be an $\mathrm{N}(0,1)$-distributed random variable and set $X(t)=Y$ for $t\in\mathbb{R}$. Further, $S_{XX}(f)=\delta(f)$ since $(\mathcal{F}^{-1}\delta)(\tau)=(\mathcal{F}\delta)(\tau)=1$.

Task 5. Symmetry gives $P(\text{decision 0 when 1 sent})=P(\text{decision 1 when 0 sent})=$ $P\big((h\star N)(t_0)>\frac{1}{2}\,(h\star s)(t_0)\big)=P\big(\mathrm{N}(0,\mathrm{Var}((h\star N)(t_0)))>\frac{1}{2}\,(h\star s)(t_0)\big)=1-\Phi(\frac{1}{2}\sqrt{\mathrm{SNR}}\,)$.

Task 6. See Tomas McKelvey's lecture slides.