MVE136 Random Signals Analysis

Written exam Tuesday 7 January 2020 2 PM–6 PM

TEACHER: Patrik Albin. JOUR: Olof Elias, telephone 031 7726457.

AIDS: Beta or 2 sheets (=4 pages) of hand-written notes (computer print-outs and/or xerox-copies are not allowed), but not both these aids.

GRADES: 12 (40%), 18 (60%) and 24 (80%) points for grade 3, 4 and 5, respectively.

MOTIVATIONS: All answers/solutions must be motivated. GOOD LUCK!

Task 1. Calculate $\Pr(X(1) = Y(2))$ when $X(t)$ and $Y(t)$, $t \in \mathbb{R}$, are two independent zero-mean WSS Gaussian processes with PSD's $S_{XX}(f) = S_{YY}(f) = \mathrm{e}^{-|f|}$. **(5 points)**

Task 2. (a) Show that the characteristic function $\Phi_X(\omega) = E(\mathrm{e}^{j\omega X})$ for a Poisson distributed random variable with mean $\lambda > 0$ is $\Phi_X(\omega) = \mathrm{e}^{\lambda(\mathrm{e}^{j\omega}-1)}$. **(2.5 points)**

(b) Show that the sum of two independent Poisson distributed random variables is a Poisson distributed random variable. **(2.5 points)**

Task 3. Let $X(n)$, $n = 0, 1, 2, \ldots$, be a Markov chain with possible values (/states) 0 and 1 and transition matrix $P = \left(\begin{smallmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{smallmatrix}\right)$. Find $\Pr(X(0) = X(2) = \ldots = X(2n) = 1)$ for $n = 0, 1, 2, \ldots$ when $X(0)$ has the stationary distribution. **(5 points)**

Task 4. Show by an example that two random processes $X(t)$ and $Y(t)$ can have common mean functions $\mu_X(t) = \mu_Y(t)$ and autocorrelation functions $R_{XX}(s,t) = R_{YY}(s,t)$ but still differ probabilistically [so that, e.g., $F_{X(t)}(z) \neq F_{Y(t)}(z)$]. **(5 points)**

Task 5. A WSS process $X(t)$ with PSD $S_{XX}(f)$ is sent on a noisy channel where an independent zero-mean WSS noise $N(t)$ with PSD $S_{NN}(f)$ is added. The recived signal $Y(t) = X(t) + N(t)$ is processed through a Wiener filter LTI system with transfer function $H(f) = S_{XX}(f)/(S_{XX}(f) + S_{NN}(f))$ and outsignal $Z(t)$ that minimizes the mean-square distance $D = E((X(t) - Z(t))^2)$ between the sent signal and the processed recived signal. Express D in terms of $S_{XX}(f)$ and $S_{NN}(f)$. **(5 points)**

Task 6. Explain how the Yule Walker equations for AR(p) processes are derived and what they can be used for. **(5 points)**

MVE136 Random Signals Analysis

Solutions to written exam 7 January 2020

Task 1. We have $\Pr(X(1)=Y(2)) = \Pr(X(1)-Y(2)=0) = \Pr(\mathrm{N}(\mu,\sigma^2)=0) = 0$ as $\mu = E(X(1)) - E(Y(2)) = 0$ and $\sigma^2 = E((X(1)-Y(2))^2) = E(X(1)^2) + E(Y(2)^2) = 2\int_{-\infty}^{\infty} \mathrm{e}^{-|f|}\, df > 0$.

Task 2. (a) $E(\mathrm{e}^{j\omega X}) = \sum_{k=0}^{\infty} \mathrm{e}^{j\omega k}\, \frac{\lambda^k}{k!}\, \mathrm{e}^{-\lambda} = \mathrm{e}^{\mathrm{e}^{j\omega}\lambda}\, \mathrm{e}^{-\lambda}$.

(b) We have $\Phi_{X+Y}(\omega) = E(\mathrm{e}^{j\omega(X+Y)}) = E(\mathrm{e}^{j\omega X})\, E(\mathrm{e}^{j\omega Y}) = \mathrm{e}^{\lambda_X(\mathrm{e}^{j\omega}-1)}\, \mathrm{e}^{\lambda_Y(\mathrm{e}^{j\omega}-1)} = \mathrm{e}^{(\lambda_X+\lambda_Y)(\mathrm{e}^{j\omega}-1)}$ so that $X+Y$ is Poisson distributed with mean $\lambda_X+\lambda_Y$.

Task 3. As $P^2 = P$ and $\pi = (1/2\ \ 1/2)$ we have $\Pr(X(0) = \ldots = X(2n) = 1) = (\pi)_1\, ((P^2)_{11})^n = 2^{-(n+1)}$.

Task 4. Take $X(t) = \xi$ for a single $\mathrm{N}(0,1)$-distributed random variable and $Y(t) = \eta$ for a random sign variable $\Pr(\eta = 1) = \Pr(\eta = -1)$. Then $\mu_X(t) = \mu_Y(t) = 0$ and $R_{XX}(s,t) = R_{YY}(s,t) = 1$ while clearly $F_{X(t)}(z) = \Phi(z) \neq F_\eta(z) = F_{Y(t)}(z)$.

Task 5. From the derivation of the Wiener filter we have

$$\begin{aligned} D &= \int_{-\infty}^{\infty} \big(|H(f)|^2 S_{XX}(f) + |H(f)|^2 S_{NN}(f) + S_{XX}(f) - 2\,H(f)\,S_{XX}(f)\big)\, df \\ &= \int_{-\infty}^{\infty} S_{XX}(f) S_{NN}(f)/(S_{XX}(f) + S_{NN}(f))\, df. \end{aligned}$$

Task 6. See Tomas McKelvey's lecture notes.